

THE ABSTRACT QUASILINEAR CAUCHY PROBLEM FOR A MEMS MODEL WITH TWO FREE BOUNDARIES

MARTIN KOHLMANN

ABSTRACT. In this paper, we reformulate a mathematical model for the dynamics of an idealized electrostatically actuated MEMS device with two elastic membranes as an initial value problem for an abstract quasilinear evolution equation. Applying the Contraction Mapping Theorem, it is shown that the model is locally well-posed in time for any value of the source voltage of the device. In addition it is proven that the MEMS model considered here possesses global solutions for small source voltages whereas for large source voltages solutions of the model have a finite maximal existence time. Furthermore, we comment on the relationship of our model to its stationary version and to its small aspect ratio limit by showing that there exists a unique exponentially stable steady state and by proving convergence towards a solution of the narrow gap model in the vanishing aspect ratio limit. Our results extend the discussion of the elliptic-parabolic MEMS model presented in [25] leading to a Cauchy problem for a semilinear abstract evolution equation.

1. INTRODUCTION AND MAIN RESULTS

Micro-electro mechanical systems (MEMS) are small devices that operate on the principle of electrostatic actuation: applying a potential difference between certain mechanical components of the device causes an electric field and hence a Coulomb force resulting in a mechanical deformation. There are a wide range of applications of MEMS to report on: MEMS are used as microsensors and microactuators, they appear as components of accelerometers and gyroscopes, they have commercial applications, e.g., in microphones and mobile phones, and Bio-MEMS are used in medical and health technology. In recent years, MEMS have also become a flourishing field of research in mathematics as various types of models for such devices have been proposed. Most often these models are concerned with an idealized device consisting of a deflecting membrane suspended above a fixed ground plate.

In the paper at hand we discuss a moving boundary problem for a so-called DFM device, i.e., a MEMS with double freestanding membranes as explained in, e.g., [12]. Our model can be derived as follows: Let $H, L > 0$ and denote by (\hat{x}, \hat{z}) coordinates of the two-dimensional rectangular domain $R = (-L, L) \times (0, -H)$. We consider two thin, conductive and elastic membranes of length $2L$ and distance H located at the upper and the lower boundary of R which should be held fixed at $(\pm L, 0)$

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and $(\pm L, -H)$. Moreover, we assume that the permittivity of the medium filling the interior of R is equal to one. When a non-zero source voltage V is applied to the device, an electric field sets up causing a deformation of the membranes whose displacements are then modeled by functions \hat{u}, \hat{v} satisfying $-H < \hat{v}(\hat{x}) < \hat{u}(\hat{x}) < 0$, for $\hat{x} \in (-L, L)$, and $(\hat{u}, \hat{v})(\pm L) = (0, -H)$. Let $\hat{\varphi}(\hat{x}, \hat{z})$ denote the electrostatic potential defined in the region

$$\hat{\Omega}_{\hat{u}, \hat{v}} := \{(\hat{x}, \hat{z}) \in R; \hat{v}(\hat{x}) < \hat{z} < \hat{u}(\hat{x})\}$$

between the membranes. Then $\hat{\varphi}$ is a solution to the Laplace equation,

$$\partial_{\hat{x}}^2 \hat{\varphi} + \partial_{\hat{z}}^2 \hat{\varphi} = 0 \quad \text{in } \hat{\Omega}_{\hat{u}, \hat{v}},$$

with the boundary conditions

$$\begin{aligned} \hat{\varphi}(\hat{x}, \hat{v}(\hat{x})) &= 0, & \hat{x} &\in (-L, L), \\ \hat{\varphi}(\hat{x}, \hat{u}(\hat{x})) &= V, & \hat{x} &\in (-L, L). \end{aligned}$$

We assume that the continuous extension of $\hat{\varphi}$ to the lateral boundary of R depends linearly on \hat{z} . The total potential energy $E(\hat{u}, \hat{v})$ of the device is the sum of the electrostatic energy determined by the square of the gradient of the potential plus the elastic energy determined by the change of the length of the elastic membranes. To be able to compare the strengths of the mechanical and electrical forces in the device, we also introduce surface tension coefficients $T_1, T_2 > 0$ so that

$$\begin{aligned} E(\hat{u}, \hat{v}) &= \frac{\varepsilon_0}{2} \int_{-L}^L \int_{\hat{v}(\hat{x})}^{\hat{u}(\hat{x})} |\nabla \hat{\varphi}(\hat{x}, \hat{z})|^2 d\hat{x} d\hat{z} + T_1 \int_{-L}^L \left(\sqrt{1 + (\partial_{\hat{x}} \hat{u}(\hat{x}))^2} - 1 \right) d\hat{x} \\ &\quad + T_2 \int_{-L}^L \left(\sqrt{1 + (\partial_{\hat{x}} \hat{v}(\hat{x}))^2} - 1 \right) d\hat{x}, \end{aligned}$$

where ε_0 is the permittivity of free space. We now define dimensionless variables

$$x = \frac{\hat{x}}{L}, \quad z = \frac{\hat{z}}{H}, \quad u = \frac{\hat{u}}{H}, \quad v = \frac{\hat{v}}{H}, \quad \varphi = \frac{\hat{\varphi}}{V}$$

and parameters

$$\varepsilon = \frac{H}{L}, \quad \lambda = \frac{\varepsilon_0 V^2}{2\varepsilon^3 T_1 L}, \quad \mu = \frac{\varepsilon_0 V^2}{2\varepsilon^3 T_2 L},$$

we introduce the sets

$$\begin{aligned} \Omega_{u,v} &= \{(x, z) \in (-1, 1) \times (-1, 0); v(x) < z < u(x)\}, \\ \Gamma_u &= \{(x, u(x)); x \in I\}, \\ \Gamma_v &= \{(x, v(x)); x \in I\}, \end{aligned}$$

shown in Figure 1, and $I = (-1, 1)$ and define the operators $\nabla_\varepsilon = (\varepsilon \partial_x, \partial_z)$ and $\Delta_\varepsilon = \varepsilon^2 \partial_x^2 + \partial_z^2$.

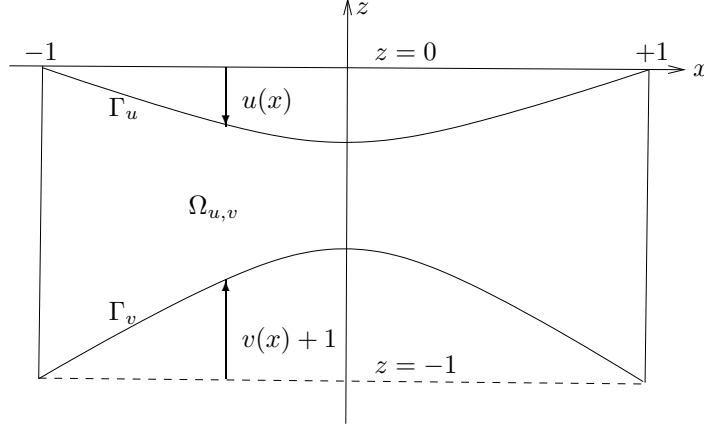


Figure 1: An idealized model for an electrostatic MEMS device with two free boundaries.

The total energy of the device can be rewritten as

$$\begin{aligned}
 E(\hat{u}, \hat{v}) &= \frac{\varepsilon_0}{2\varepsilon} V^2 \int_{\Omega_{u,v}} |\nabla_\varepsilon \varphi(x, z)|^2 dx dz + T_1 L \int_I \left(\sqrt{1 + \varepsilon^2 (\partial_x u)^2} - 1 \right) dx \\
 &\quad + T_2 L \int_I \left(\sqrt{1 + \varepsilon^2 (\partial_x v)^2} - 1 \right) dx \\
 &= \int_I \mathcal{L}(x, u, v, \partial_x u, \partial_x v) dx
 \end{aligned}$$

with \mathcal{L} denoting the Lagrangian density. The Euler-Lagrange equations $\partial_x \partial_{\partial_x u} \mathcal{L} - \partial_u \mathcal{L} = 0$ and $\partial_x \partial_{\partial_x v} \mathcal{L} - \partial_v \mathcal{L} = 0$ take the form

$$\begin{aligned}
 \partial_x \left(\frac{\partial_x u}{\sqrt{1 + \varepsilon^2 (\partial_x u)^2}} \right) - \lambda |\nabla_\varepsilon \varphi(x, u(x))|^2 &= 0, \\
 \partial_x \left(\frac{\partial_x v}{\sqrt{1 + \varepsilon^2 (\partial_x v)^2}} \right) + \mu |\nabla_\varepsilon \varphi(x, v(x))|^2 &= 0.
 \end{aligned}$$

We now assume that u and v also depend on time \hat{t} . Then $\partial_{\hat{t}}^2 u$ models the acceleration of Γ_u and Γ_v in the associated evolution problem. Regarding the left-hand sides of the Euler-Lagrange equations above as forces on Γ_u and Γ_v and considering a damping force that is proportional to the velocity $\partial_{\hat{t}} u$, Newton's Second Law yields that

$$\begin{aligned}
 \rho_1 \delta_1 \partial_{\hat{t}}^2 u + a \partial_{\hat{t}} u &= \partial_x \left(\frac{\partial_x u}{\sqrt{1 + \varepsilon^2 (\partial_x u)^2}} \right) - \lambda |\nabla_\varepsilon \varphi(x, u(x))|^2, \\
 \rho_2 \delta_2 \partial_{\hat{t}}^2 v + a \partial_{\hat{t}} v &= \partial_x \left(\frac{\partial_x v}{\sqrt{1 + \varepsilon^2 (\partial_x v)^2}} \right) + \mu |\nabla_\varepsilon \varphi(x, v(x))|^2,
 \end{aligned}$$

where ρ_1, ρ_2 and δ_1, δ_2 denote the mass density per unit volume of the membranes and the membrane thicknesses respectively and a is a damping constant. With

$$t = \frac{\hat{t}}{a}, \quad \gamma_1 = \frac{\sqrt{\rho_1 \delta_1}}{a}, \quad \gamma_2 = \frac{\sqrt{\rho_2 \delta_2}}{a}$$

we arrive at the non-dimensionalized equations

$$\begin{aligned} \gamma_1^2 \partial_t^2 u + \partial_t u &= \partial_x \left(\frac{\partial_x u}{\sqrt{1 + \varepsilon^2 (\partial_x u)^2}} \right) - \lambda |\nabla_\varepsilon \varphi(x, u(x))|^2, \\ \gamma_2^2 \partial_t^2 v + \partial_t v &= \partial_x \left(\frac{\partial_x v}{\sqrt{1 + \varepsilon^2 (\partial_x v)^2}} \right) + \mu |\nabla_\varepsilon \varphi(x, v(x))|^2. \end{aligned}$$

In this paper, we will assume that $\gamma_1, \gamma_2 \ll 1$ meaning that the damping forces dominate over the inertial forces. Given initial values u_0 and v_0 for the functions u and v , we thus discuss the following system of equations:

$$\begin{aligned} (1) \quad & -\Delta_\varepsilon \varphi = 0, & \text{in } \Omega_{u,v}, t > 0, \\ (2) \quad & \varphi = \frac{z - v}{u - v}, & \text{on } \partial\Omega_{u,v}, t > 0, \\ (3) \quad & \partial_t u - \partial_x \left(\frac{\partial_x u}{\sqrt{1 + \varepsilon^2 (\partial_x u)^2}} \right) = -\lambda |\nabla_\varepsilon \varphi|^2, & \text{on } \Gamma_u, t > 0, \\ (4) \quad & \partial_t v - \partial_x \left(\frac{\partial_x v}{\sqrt{1 + \varepsilon^2 (\partial_x v)^2}} \right) = \mu |\nabla_\varepsilon \varphi|^2, & \text{on } \Gamma_v, t > 0, \\ (5) \quad & u(t, \pm 1) = 0, & t > 0, \\ (6) \quad & v(t, \pm 1) = -1, & t > 0, \\ (7) \quad & u(0, x) = u_0, & x \in I, \\ (8) \quad & v(0, x) = v_0, & x \in I. \end{aligned}$$

Note that (1)–(8) is a free boundary problem as the domain $\Omega_{u,v}$ and its boundary components Γ_u, Γ_v have to be determined together with the solution (u, v, φ) . Several simplified models of (1)–(8) have been studied recently: In [25] we have assumed that the deformation of the membranes is small so that, in the equations on the free boundaries, the curvature terms on the left-hand sides of (3)–(4) can be replaced by the linear terms $-\partial_x^2 u$ and $-\partial_x^2 v$. In this case, the evolution of the membranes is described by two heat equations with a right-hand side proportional to the square of the gradient of the potential on the boundary. In [24] the stationary version of the MEMS model with two free boundaries and linear stretching terms has been discussed. For $v \equiv -1$, the problem (1)–(8) models the evolution of a free membrane suspended above a fixed ground plate. Various analytical results on this type of a MEMS have been obtained in recent years: [8, 14, 16, 19, 20, 22, 27, 28] refer to the parabolic problem, [6, 13, 21, 23, 27] discuss the problem with a hyperbolic evolution equation and in [7, 26, 31–33] the stationary model is presented. The corresponding model with an additional curvature term is discussed in [9] and our derivation of (1)–(8) refines Laurençot’s line of arguments therein.

For $\varepsilon \rightarrow 0$, one obtains the so-called small aspect ratio model from (1)–(8):

$$\begin{aligned}
(9) \quad & \varphi = \frac{z - v}{u - v}, & \text{in } \Omega_{u,v} \cup \partial\Omega_{u,v}, \quad t > 0, \\
(10) \quad & \partial_t u - \partial_x^2 u = -\frac{\lambda}{(u - v)^2}, & x \in I, \quad t > 0, \\
(11) \quad & \partial_t v - \partial_x^2 v = \frac{\mu}{(u - v)^2}, & x \in I, \quad t > 0, \\
(12) \quad & u(t, \pm 1) = 0, & t > 0, \\
(13) \quad & v(t, \pm 1) = -1, & t > 0, \\
(14) \quad & u(0, x) = u_0, & x \in I, \\
(15) \quad & v(0, x) = v_0, & x \in I.
\end{aligned}$$

The problem (9)–(15) already appeared in [25] where we proved that solutions of the MEMS model of [25] with $\varepsilon > 0$ converge towards solutions of (9)–(15) in the vanishing aspect ratio limit. The small aspect ratio limit of the MEMS model with a fixed ground plate is a subject of [5, 9, 11, 13–16, 18–20, 22, 29, 34].

The plan of the present paper is to apply and refine the chain of arguments used in [9, 25] in order to obtain results on solutions of (1)–(8) where we have to cope with additional curvature terms compared to the model in [25]. In doing so, our first aim is to show that (1)–(8) possesses a unique maximal solution for any pair of values (λ, μ) . To this end, we solve the elliptic problem (1)–(2) for the potential and then rewrite the system (3)–(8) as an initial value problem for an abstract quasilinear evolution equation whose solution is obtained from the variation of constants formula and the Contraction Mapping Theorem. Our main endeavour is to prove the Lipschitz continuity of the right-hand side with respect to the topology of $W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)$, $\xi > 0$; see also the semilinear problems in [8, 25] where this has been achieved for $\xi = 0$. Our first main result which is the analog of [25, Theorem 2] and [9, Theorem 1.1] reads as follows.

Theorem 1. *Let $q \in (2, \infty)$ and $\varepsilon \in (0, 1)$ and consider initial values $(u_0, v_0) \in W_q^2(I) \times W_q^2(I)$ satisfying $(u_0, v_0)(\pm 1) = (0, -1)$ and $-1 \leq v_0(x) < u_0(x) \leq 0$ for all $x \in I$. Then:*

- (i) *There exists $r > 0$ such that for $\|u_0\|_{W_q^2(I)}, \|v_0 + 1\|_{W_q^2(I)} < r$ and for any $\lambda, \mu > 0$, there is a unique maximal solution $(u, v, \varphi)(t)$, $t \in [0, T_\varepsilon)$, $T_\varepsilon > 0$, to (1)–(8) with regularity*

$$u, v \in C([0, T_\varepsilon), W_q^2(I)) \cap C^1([0, T_\varepsilon), L_q(I)), \quad \varphi \in W_2^2(\Omega_{u(t), v(t)})$$

so that $-1 \leq v < u \leq 0$ on $[0, T_\varepsilon) \times I$.

- (ii) *If for each $\tau > 0$ there exists $\kappa(\tau) \in (0, \frac{1}{2})$ such that $u(t) - v(t) \geq 2\kappa(\tau)$ and $\|u(t)\|_{W_q^2(I)}, \|v(t) + 1\|_{W_q^2(I)} \leq \kappa(\tau)^{-1}$ for $t \in [0, T_\varepsilon) \cap [0, \tau]$, then the solution exists globally in time, i.e., $T_\varepsilon = \infty$.*
- (iii) *If u_0 and v_0 are even functions on I , then (u, v, φ) is even in x on $[0, T_\varepsilon) \times I$.*
- (iv) *Given $\kappa \in (0, \frac{1}{2})$, there exist $m_0(\kappa), r_0(\kappa) > 0$ such that, for $\max\{\lambda, \mu\} < m_0(\kappa)$ and $\|u_0\|_{W_q^2(I)}, \|v_0 + 1\|_{W_q^2(I)} < r_0(\kappa)$, one has $T_\varepsilon = \infty$, $u(t) - v(t) \geq$*

2κ and $\|u(t)\|_{W_q^2(I)}$ and $\|v(t) + 1\|_{W_q^2(I)}$ are bounded by a positive constant only depending on κ .

Observe that, in contrast to Theorem 2 of [25] and Theorem 1.1 of [9], we have to assume that the initial values (u_0, v_0) are sufficiently small in $W_q^2(I) \times W_q^2(I)$ here. A proof of Theorem 1 can be found in Section 2. The methods used in Section 2 also yield that solutions of (1)–(8) converge towards a solution of (9)–(15) for $\varepsilon \rightarrow 0$. We present a proof of the following theorem which is the analog of [25, Theorem 10] and [9, Theorem 1.4] and which justifies rigorously the relationship between the original problem and its small aspect ratio limit. Here, $\mathbf{1}_A$ denotes the indicator function of the set $A \subset \mathbb{R}^2$.

Theorem 2. *Let $\lambda, \mu > 0$, $q \in (2, \infty)$ and let $(u_0, v_0) \in W_q^2(I) \times W_q^2(I)$ satisfying the assumptions in Theorem 1 be given. For $\varepsilon \in (0, 1)$, the unique solution to (1)–(8) with initial values (u_0, v_0) and the maximal interval of existence $[0, T_\varepsilon)$ is denoted by $(u_\varepsilon, v_\varepsilon, \varphi_\varepsilon)(t)$. Then there are $\tau > 0$, $\varepsilon_* \in (0, 1)$ and $\kappa_1 \in (0, \frac{1}{2})$ such that $T_\varepsilon \geq \tau$, $u_\varepsilon(t) - v_\varepsilon(t) \geq 2\kappa_1$ and $\|u_\varepsilon(t)\|_{W_q^2(I)}, \|v_\varepsilon(t) + 1\|_{W_q^2(I)} \leq \kappa_1^{-1}$, for all $(t, \varepsilon) \in [0, \tau] \times (0, \varepsilon_*)$. Moreover, the small aspect ratio model (9)–(15) has a unique solution (u_*, v_*, φ_*) satisfying*

$$u_*, v_* \in C([0, \tau], W_q^2(I)) \cap C^1([0, \tau], L_q(I)),$$

$-1 \leq v_*(t) < u_*(t) \leq 0$, $u_*(t) - v_*(t) \geq 2\kappa_1$, for all $t \in [0, \tau]$, and there is a null sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subset (0, \varepsilon_*)$ such that

$$\begin{aligned} (u_{\varepsilon_n}, v_{\varepsilon_n}) &\rightarrow (u_*, v_*) && \text{in } C^{1-\theta}([0, \tau], W_q^{2\theta}(I)), \theta \in (0, 1), \\ \varphi_{\varepsilon_n}(t) \mathbf{1}_{\Omega_{u_{\varepsilon_n}(t), v_{\varepsilon_n}(t)}} &\rightarrow \varphi_*(t) \mathbf{1}_{\Omega_{u_*(t), v_*(t)}} && \text{in } L_2(I \times (-1, 0)), t \in [0, \tau], \end{aligned}$$

as $n \rightarrow \infty$. Furthermore, there is $\Lambda(\kappa) > 0$ such that, for $\lambda, \mu < \Lambda(\kappa)$, the statements of the theorem hold true for any $\tau > 0$.

In particular, Theorem 2 guarantees that the maximal existence times T_ε are bounded from below when sending $\varepsilon \rightarrow 0$. Again, in contrast to the models discussed in [25] and [9], an additional condition on the norm of the initial values occurs in the above theorem.

The effectiveness of our MEMS device is limited when increasing the source voltage as the membranes might come close and closer and finally touch. This phenomenon is called pull-in stability and has already been discussed for related models, see, e.g., [4, 8–10, 13, 14, 20, 26, 32]. It is plausible to expect that for small voltage values the problem (1)–(8) has a global solution and that for λ and μ sufficiently large, there is no steady state of (1)–(8). Recall that Theorem 1.(iv) implies that solutions $(u, v, \varphi)(t)$ exist globally in time in the sense that neither touchdown of the membranes nor blow up of the displacements in $W_q^2(I) \times W_q^2(I)$ occurs, provided λ and μ and the initial values are sufficiently small. Next, we complement this result by a non-existence theorem for high voltages. We will concentrate on displacements u and v that have a positive distance to $\{z = -1\}$ and $\{z = 0\}$ respectively, as touchdown on $\{z = -1\}$ or $\{z = 0\}$ is reminiscent of the associated

MEMS problem with only one free membrane. For sufficiently large values λ and μ , we divine that $T_\varepsilon < \infty$ and

$$(16) \quad \limsup_{t \rightarrow T_\varepsilon} \|(u, v)(t)\|_{W_q^2(I) \times W_q^2(I)} = \infty \quad \text{or} \quad \liminf_{t \rightarrow T_\varepsilon} \min\{u(t) - v(t)\} = 0.$$

It will remain an open problem whether the membranes certainly smash-up when $T_\varepsilon < \infty$. By (16), the displacements might also blow up in $W_q^2(I) \times W_q^2(I)$ contradicting the physical expectation that there is collision of the membranes in the interior of the device for finite maximal existence times. A similar ambiguity has been observed in [27]. In Section 3 we present proofs of the following theorems.

Theorem 3. *Let $q \in (2, \infty)$ and $\varepsilon \in (0, 1)$. There exists a positive number $\xi_0(\varepsilon)$ such that for $\max\{\lambda, \mu\} > \xi_0(\varepsilon)$ the stationary problem (1)–(8) possesses no steady state solution (u, v, φ) with $u, v \in W_q^2(I)$ and $\varphi \in W_2^2(\Omega_{u,v})$ satisfying $-1 \leq v < u \leq 0$ on I . In addition $\xi_0(\varepsilon) \rightarrow 2$ for $\varepsilon \rightarrow 0$.*

Theorem 4. *Let $q \in (2, \infty)$ and $u_0, v_0 \in W_q^2(I)$ satisfying the assumptions in Theorem 1 and the additional assumption $(u_0, v_0)(-x) = (u_0, v_0)(x)$, for all $x \in I$, be given. There exists $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds true: If $\max\{\lambda, \mu\} > 4/\varepsilon$, the displacements (u, v) do not blow up in $W_q^2(I) \times W_q^2(I)$ and $v, -u - 1 \leq c$, for some $c < 0$, then the maximal existence time of the solution (u, v, φ) obtained in Theorem 1 is finite and*

$$(17) \quad T_\varepsilon \leq \frac{\|u_0 - v_0\|_{L_1(I)}}{\max\{\lambda, \mu\} - 4/\varepsilon}.$$

If equality holds in (17), there is touchdown of the membranes in the sense that $\liminf_{t \rightarrow T_\varepsilon} \min\{u(t) - v(t)\} = 0$.

Note that Theorem 3 and Theorem 4 are the analogs of [25, Theorem 5], [9, Theorem 1.3] and [10, Theorem 3].

Finally, in Section 4, it will be established that, for any $\kappa \in (0, 1/2)$, (1)–(8) possesses a unique steady state so that the boundary components have distance at least 2κ and the $W_q^2(I)$ -norms of the first and second component are bounded by κ^{-1} . Moreover exponential stability of this steady state is shown using the Principle of Linearized Stability. The following theorem is the analog of [25, Theorem 6] and [9, Theorem 1.2].

Theorem 5. *Let $q \in (2, \infty)$, $\varepsilon \in (0, 1)$ and $\kappa \in (0, \frac{1}{2})$ be fixed.*

- (i) *There are $\delta(\kappa) > 0$ and an analytic function $[0, \delta)^2 \rightarrow W_q^2(I) \times W_q^2(I)$, $\Lambda \rightarrow U_\Lambda = (U_{\Lambda,1}, U_{\Lambda,2})$, such that $(U_\Lambda, \Phi_\Lambda)$ is for each $\Lambda = (\lambda, \mu) \in (0, \delta)^2$ the unique steady state of (1)–(8) satisfying $U_{\Lambda,1} - U_{\Lambda,2} \geq 2\kappa$ and $\|U_{\Lambda,1}\|_{W_q^2(I)}, \|U_{\Lambda,2} + 1\|_{W_q^2(I)} \leq \kappa^{-1}$ and $\Phi_\Lambda \in W_2^2(\Omega_{U_{\Lambda,1}, U_{\Lambda,2}})$ is the potential associated with U_Λ . Moreover, $U_{\Lambda,1}$ and $-U_{\Lambda,2}$ are convex and even with $U_{(0,0)} = (0, 0)$ and $x \mapsto \Phi_\Lambda(x, z)$ is even on I .*
- (ii) *Let $\Lambda \in (0, \delta)^2$. There are numbers $\omega_0, \varrho, R > 0$ such that for each pair of initial values $u_0, v_0 \in W_q^2(I)$ satisfying the assumptions in Theorem 1 and*

the additional assumption $\|(u_0, v_0) - U_\Lambda\|_{W_q^2(I) \times W_q^2(I)} < \varrho$, the associated solution (u, v, φ) to (1)–(8) exists globally in time with $u(t) - v(t) > 0$ and

$$\begin{aligned} & \|(u, v) - U_\Lambda\|_{W_q^2(I) \times W_q^2(I)} + \|(u_t, v_t)\|_{L_q(I) \times L_q(I)} \\ & \leq Re^{-\omega_0 t} \|(u_0, v_0) - U_\Lambda\|_{W_q^2(I) \times W_q^2(I)}, \quad \forall t \geq 0. \end{aligned}$$

A convergence result similar to Theorem 5.(ii) holds true for the first component φ of the solution, cf. Section 4 for the technical details.

2. LOCAL AND GLOBAL WELL-POSEDNESS AND THE SMALL ASPECT RATIO LIMIT

In this section, we present proofs of Theorem 1 and Theorem 2. Let us first introduce our notation and recall some preliminary results: Let $\Omega := I \times (0, 1)$ and consider the time-dependent transformation of coordinates $T = T_{u(t), v(t)} : \Omega_{u(t), v(t)} \rightarrow \Omega$ given by

$$T(x, z) = (x', z') = \left(x, \frac{z - v(t, x)}{u(t, x) - v(t, x)} \right).$$

With the definition of $\Omega_{u(t), v(t)}$ in Section 1, it is easily checked that $T_{u(t), v(t)}$ is a diffeomorphism $\Omega_{u(t), v(t)} \rightarrow \Omega$ with the inverse

$$T^{-1}(x', z') = (x', z'(u(t, x') - v(t, x')) + v(t, x'))$$

and it is clear that T and T^{-1} can be extended to the boundary of $\Omega_{u(t), v(t)}$ and Ω respectively. We introduce pull-back and push-forward operators $\theta^*(u, v)$ and $\theta_*(u, v)$ defined by $\theta^*(u, v)\tilde{w} = \tilde{w} \circ T_{u, v}$ and $\theta_*(u, v)w = w \circ T_{u, v}^{-1}$ where w and \tilde{w} are functions of the coordinates (x, z) and (x', z') respectively, i.e.,

$$[\theta^*(u, v)\tilde{w}](x, z) = \tilde{w}(T_{u, v}(x, z)) \quad \text{and} \quad [\theta_*(u, v)w](x', z') = w(T_{u, v}^{-1}(x', z')).$$

Let $\tilde{\Delta}_{u, v; \varepsilon} = \theta_*(u, v)\Delta_\varepsilon\theta^*(u, v)$ denote the time-dependent transformed Laplace operator on Ω which is explicitly given by

$$\begin{aligned} \tilde{\Delta}_{u, v; \varepsilon}\tilde{w} &= \varepsilon^2 \tilde{w}_{x'x'} - 2\varepsilon^2 \tilde{w}_{x'z'} \frac{z'(u_{x'} - v_{x'}) + v_{x'}}{u - v} + \tilde{w}_{z'z'} \frac{1 + \varepsilon^2 [z'(u_{x'} - v_{x'}) + v_{x'}]^2}{(u - v)^2} \\ &+ \varepsilon^2 \tilde{w}_{z'} \left(2 \frac{u_{x'} - v_{x'}}{(u - v)^2} [z'(u_{x'} - v_{x'}) + v_{x'}] - \frac{z'(u_{x'x'} - v_{x'x'}) + v_{x'x'}}{u - v} \right); \end{aligned}$$

here the notation $u_{x'}$ stands for $\partial_{x'}u$ et cetera.

For $q \in (2, \infty)$, we introduce the function spaces

$$W_{q, D}^{2\alpha}(I) := \begin{cases} \{w \in W_q^{2\alpha}(I); w(\pm 1) = 0\}, & 2\alpha \in (1/q, 2], \\ W_q^{2\alpha}(I), & 0 \leq 2\alpha < 1/q, \end{cases}$$

and we define

$$W_{2, D}^\alpha(\Omega) := \begin{cases} \{w \in W_2^\alpha(\Omega); u|_{\partial\Omega} = 0\}, & \alpha > 1/2, \\ W_2^\alpha(\Omega), & 0 \leq \alpha < 1/2; \end{cases}$$

the index D indicates the Dirichlet boundary condition. The space $W_{2, D}^1(\Omega)$ is equipped with the norm $\|w\|_{W_{2, D}^1(\Omega)} = \|\nabla w\|_{L_2(\Omega)}$ and we will use the notation $W_{2, D}^{-\alpha}(\Omega)$ for the dual space $(W_{2, D}^\alpha(\Omega))'$, for $0 \leq \alpha \leq 1$.

For $q \in (2, \infty)$ and $\kappa \in (0, 1/2)$, we define the sets

$$S_q(\kappa) := \left\{ (u, v) \in W_q^2(I) \times W_q^2(I); (u, v)(\pm 1) = (0, -1), \|u\|_{W_{q,D}^2(I)} < \frac{1}{\kappa}, \right. \\ \left. \|v + 1\|_{W_{q,D}^2(I)} < \frac{1}{\kappa}, 2\kappa < u(x) - v(x), \forall x \in I \right\}$$

and prepare the following lemma.

Lemma 6. *The sets $S_q(\kappa) + \{(0, 1)\} \subset W_{q,D}^2(I) \times W_{q,D}^2(I)$ are open for $q \in (2, \infty)$ and $\kappa \in (0, 1/2)$ and the closure of $S_q(\kappa)$ denoted as $\bar{S}_q(\kappa)$ is given by*

$$\bar{S}_q(\kappa) = \left\{ (u, v) \in W_q^2(I) \times W_q^2(I); (u, v)(\pm 1) = (0, -1), \|u\|_{W_{q,D}^2(I)} \leq \frac{1}{\kappa}, \right. \\ (18) \quad \left. \|v + 1\|_{W_{q,D}^2(I)} \leq \frac{1}{\kappa}, 2\kappa \leq u(x) - v(x), \forall x \in I \right\}.$$

Proof. For $q \in (2, \infty)$ and $\kappa \in (0, 1/2)$ given, let $\tilde{S}_q(\kappa)$ be the set $S_q(\kappa) + \{(0, 1)\}$. Then $(u, \tilde{v}) \in \tilde{S}_q(\kappa)$ if and only if $(u, \tilde{v} - 1) \in S_q(\kappa)$ which is equivalent to

- (1) $u, \tilde{v} - 1 \in W_q^2(I)$,
- (2) $(u, \tilde{v})(\pm 1) = (0, 0)$,
- (3) $\|u\|_{W_q^2(I)}, \|\tilde{v}\|_{W_q^2(I)} < 1/\kappa$,
- (4) $2\kappa < u - \tilde{v} + 1$ on I .

The lemma claims that given $(u, \tilde{v}) \in \tilde{S}_q(\kappa)$ there exists $\varepsilon > 0$ such that

$$(u + \varepsilon w_1, \tilde{v} + \varepsilon w_2) \in \tilde{S}_q(\kappa)$$

for $w_1, w_2 \in W_{q,D}^2(I)$, $\|w_1\|_{W_q^2(I)}, \|w_2\|_{W_q^2(I)} < 1$. To prove this, we first note that due to Sobolev's embedding theorem $W_q^2(I) \hookrightarrow C^1(I)$, so that there is a constant $c > 0$ only depending on q such that

$$\|w_1\|_\infty, \|w_{1x}\|_\infty, \|w_2\|_\infty, \|w_{2x}\|_\infty \leq c.$$

Now we observe:

- (1) As $W_q^2(I)$ is a vector space, it is clear that $u + \varepsilon w_1$ and $\tilde{v} + \varepsilon w_2 - 1 = (\tilde{v} - 1) + \varepsilon w_2$ belong to $W_q^2(I)$ for $(u, \tilde{v} - 1) \in W_q^2(I)$ and $w_1, w_2 \in W_{q,D}^2(I)$.
- (2) As w_1, w_2 have Dirichlet boundary conditions on $[-1, 1]$ it is also clear that $(u + \varepsilon w_1, \tilde{v} + \varepsilon w_2)(\pm 1) = (0, 0)$.
- (3) There exists $\delta_1 > 0$ such that $\|u\|_{W_q^2(I)} \leq 1/\kappa - \delta_1$. Then

$$\|u + \varepsilon w_1\|_{W_q^2(I)} < 1/\kappa + \varepsilon - \delta_1 < 1/\kappa$$

for $\varepsilon < \delta_1$. The corresponding estimate for \tilde{v} is obtained similarly for ε smaller than a number $\delta_2 > 0$.

- (4) There exists $\delta_3 > 0$ such that $u - \tilde{v} + 1 \geq 2\kappa + \delta_3$ on I . Then

$$u + \varepsilon w_1 - (\tilde{v} + \varepsilon w_2) + 1 = u - \tilde{v} + 1 + \varepsilon(w_1 - w_2) \\ \geq 2\kappa + \delta_3 - 2c\varepsilon \\ > 2\kappa$$

for $\varepsilon < \delta_3/(2c)$.

Finally taking ε to be smaller than $\min\{\delta_1, \delta_2, \delta_3/(2c)\}$ achieves the proof of the first statement. Given a sequence $(u_n, v_n) \in S_q(\kappa)$ that converges to some $(u, v) \in W_q^2(I) \times W_q^2(I)$, it is immediately clear that (u, v) belongs to the set on the right-hand side of (18). It is an elementary proof that replacing some $<$ by \leq in the definition of $S_q(\kappa)$ yields a subsets of the closure $\overline{S}_q(\kappa)$ with respect to $W_q^2(I) \times W_q^2(I)$. This achieves the proof of the lemma. \square

2.1. The elliptic problem. We let $\tilde{\varphi}(t, x', z') = \theta_*(u(t), v(t))\varphi$, $\psi(t, x', z') = \tilde{\varphi}(t, x', z') - z'$ and

$$f_{u,v;\varepsilon} = \tilde{\Delta}_{u,v;\varepsilon} z' = \varepsilon^2 \left(2 \frac{u_{x'} - v_{x'}}{(u-v)^2} [z'(u_{x'} - v_{x'}) + v_{x'}] - \frac{z'(u_{x'x'} - v_{x'x'}) + v_{x'x'}}{u-v} \right)$$

and rewrite the elliptic problem (1)–(2) as

$$\begin{aligned} - \left(\tilde{\Delta}_{u(t), v(t);\varepsilon} \psi \right) (t, x', z') &= f_{u(t), v(t);\varepsilon}, & (x', z') \in \Omega, \ t > 0, \\ \psi(t, x', z') &= 0, & (x', z') \in \partial\Omega, \ t > 0. \end{aligned}$$

For $q \in (2, \infty)$, $\kappa \in (0, 1/2)$ and $(u, v) \in \overline{S}_q(\kappa)$, the operator $-\tilde{\Delta}_{u,v;\varepsilon}$ is elliptic with an ellipticity constant independent of (u, v) and we have the following lemma which generalizes [9, Lemma 2.2].

Lemma 7. *For each $(u, v) \in \overline{S}_q(\kappa)$ and $F \in W_{2,D}^{-1}(\Omega)$, there is a unique solution $\Phi \in W_{2,D}^1(\Omega)$ to the boundary value problem*

$$\begin{aligned} -\tilde{\Delta}_{u,v;\varepsilon} \Phi &= F, & \text{in } \Omega, \\ \Phi &= 0, & \text{on } \partial\Omega, \end{aligned}$$

and there is a constant $C_1 > 0$ only depending on κ and ε such that

$$\|\Phi\|_{W_{2,D}^1(\Omega)} \leq C_1 \|F\|_{W_{2,D}^{-1}(\Omega)}.$$

Furthermore, if $F \in L_2(\Omega)$, then $\Phi \in W_{2,D}^2(\Omega)$ and

$$\|\Phi\|_{W_{2,D}^2(\Omega)} \leq C_1 \|F\|_{L_2(\Omega)}.$$

Proof. A careful observation shows that it suffices to establish the existence of positive constants c_1, c_2 , only depending on κ and ε , such that

$$(19) \quad \|\Phi\|_{W_{2,D}^1(\Omega)} \leq c_1(\kappa, \varepsilon) \|\Phi\|_{L_2(\Omega)} + c_2(\kappa, \varepsilon) \|F\|_{W_{2,D}^{-1}(\Omega)}$$

for any test function Φ in the weak formulation of the Dirichlet problem $-\tilde{\Delta}_{u,v;\varepsilon} \Phi = F$ on Ω . Using the divergence form of $-\tilde{\Delta}_{u,v;\varepsilon}$, integration by parts and the Dirichlet boundary condition for Φ , we obtain

$$\begin{aligned} \langle F, \Phi \rangle &= \int_{\Omega} \left[\varepsilon^2 \left(\Phi_{x'} - \frac{z'(u_{x'} - v_{x'}) + v_{x'}}{u-v} \Phi_{z'} \right)^2 + \frac{\Phi_{z'}^2}{(u-v)^2} \right] dx' dz' \\ &\quad + \varepsilon^2 \int_{\Omega} \frac{u_{x'} - v_{x'}}{u-v} \left[\frac{z'(u_{x'} - v_{x'}) + v_{x'}}{u-v} \Phi_{z'} - \Phi_{x'} \right] \Phi dx' dz' \end{aligned}$$

so that, setting

$$\xi = \frac{z'(u_{x'} - v_{x'}) + v_{x'}}{u-v},$$

we get

$$(20) \quad \int_{\Omega} \left[\varepsilon^2 (\Phi_{x'} - \xi \Phi_{z'})^2 + \frac{\Phi_{z'}^2}{(u-v)^2} \right] dx' dz' \leq \|F\|_{W_{2,D}^{-1}(\Omega)} \|\Phi\|_{W_{2,D}^1(\Omega)} \\ + \varepsilon^2 \left\| \frac{u_{x'} - v_{x'}}{u-v} \right\|_{L_{\infty}(I)} \|\Phi_{x'} - \xi \Phi_{z'}\|_{L_2(\Omega)} \|\Phi\|_{L_2(\Omega)}.$$

Using $u - v \geq 2\kappa$ and $\|u\|_{C^1([-1,1])}, \|v\|_{C^1([-1,1])} \leq c_0(q, \kappa)$, an elementary computation shows that there exists a constant $0 < \nu(\kappa, \varepsilon) < 1/2$ such that for any $(\zeta_1, \zeta_2) \in \mathbb{R}^2$

$$(21) \quad \nu(\kappa, \varepsilon)(\zeta_1^2 + \zeta_2^2) \leq \varepsilon^2 (\zeta_1 - \xi \zeta_2)^2 + \frac{\zeta_2^2}{(u-v)^2}.$$

Letting $\zeta_1 = \Phi_{x'}(x', z')$ and $\zeta_2 = \Phi_{z'}(x', z')$ in (21) and integrating the inequality over Ω , we can apply the resulting estimate twice to deduce from (20) that (19) holds true with $c_1 = \varepsilon c_0/(\kappa\sqrt{\nu})$ and $c_2 = \nu^{-1}$. The existence of a unique solution $\Phi \in W_{2,D}^1(\Omega)$ satisfying the estimates stated in the lemma now follows analogously to what has been done in [9, Lemma 2.2]. \square

An immediate consequence of Lemma 7 is that the transformed problem (1)–(2) on the fixed domain Ω has a unique solution $\tilde{\varphi}_{u,v;\varepsilon} \in W_2^2(\Omega)$ satisfying

$$(22) \quad -\left(\tilde{\Delta}_{u,v;\varepsilon}\tilde{\varphi}\right)(x', z') = 0, \quad (x', z') \in \Omega,$$

$$(23) \quad \tilde{\varphi}(x', z') = z', \quad (x', z') \in \partial\Omega.$$

It is clear that, with the definition $(\tilde{u}, \tilde{v})(x) = (u, v)(-x)$, $x \in I$, we have that $\tilde{\varphi}_{\tilde{u}, \tilde{v}; \varepsilon}(x', z') = \tilde{\varphi}_{u, v; \varepsilon}(-x', z')$, $(x', z') \in \Omega$.

Henceforth, we fix $\varepsilon > 0$ and omit it as an index to simplify notation. For $(u, v) \in \overline{S}_q(\kappa)$, let us define a second order linear operator $\mathcal{A}(u, v) \in \mathcal{L}(W_{2,D}^1(\Omega), W_{2,D}^{-1}(\Omega))$ by setting

$$\mathcal{A}(u, v)\Phi = -\tilde{\Delta}_{u,v}\Phi, \quad \Phi \in W_{2,D}^1(\Omega).$$

A further consequence of Lemma 7 is that $\mathcal{A}(u, v)$ is invertible and it follows from the same arguments as in [9, Lemma 2.3] that, for all $\theta \in [0, 1] \setminus \{1/2\}$,

$$(24) \quad \|\mathcal{A}(u, v)^{-1}\|_{\mathcal{L}(W_{2,D}^{\theta-1}, W_{2,D}^{\theta+1}(\Omega))} \leq C_2(\kappa, \varepsilon), \quad \forall (u, v) \in \overline{S}_q(\kappa).$$

We now show that $\tilde{\varphi}_{u,v}$ depends Lipschitz continuously on $(u, v) \in \overline{S}_q(\kappa)$ in a suitable topology.

Lemma 8. *Given $\xi \in [0, (q-1)/q]$ and $\alpha \in (\xi, 1)$ there exists $C_3 = C_3(\kappa, \varepsilon) > 0$ so that, for all $(u_1, v_1), (u_2, v_2) \in \overline{S}_q(\kappa)$,*

$$\|\tilde{\varphi}_{u_1, v_1} - \tilde{\varphi}_{u_2, v_2}\|_{W_{2,D}^{2-\alpha}(\Omega)} \leq C_3 \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)}.$$

Proof. Since

$$\tilde{\varphi}_{u_1, v_1} - \tilde{\varphi}_{u_2, v_2} = \psi_{u_1, v_1} - \psi_{u_2, v_2} = \mathcal{A}(u_1, v_1)^{-1} f_{u_1, v_1} - \mathcal{A}(u_2, v_2)^{-1} f_{u_2, v_2}$$

and $f_{u,v} \in L_2(\Omega) \hookrightarrow W_{2,D}^{-\alpha}(\Omega)$, for all $(u, v) \in \overline{S}_q(\kappa)$, the desired estimate follows immediately from the estimates

(25)

$$\|\mathcal{A}(u_1, v_1) - \mathcal{A}(u_2, v_2)\|_{\mathcal{L}(W_{2,D}^2(\Omega), W_{2,D}^{-\alpha}(\Omega))} \leq c_1 \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)},$$

(26)

$$\|f_{u_1, v_1} - f_{u_2, v_2}\|_{W_{2,D}^{-\alpha}(\Omega)} \leq c_2 \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)},$$

(27)

$$\|\mathcal{A}(u_1, v_1)^{-1}\|_{\mathcal{L}(W_{2,D}^{-\alpha}(\Omega), W_{2,D}^{2-\alpha}(\Omega))} \leq c_3,$$

(28)

$$\|\mathcal{A}(u_2, v_2)^{-1}\|_{\mathcal{L}(L_2(\Omega), W_{2,D}^2(\Omega))} \leq c_4,$$

where c_1, \dots, c_4 are positive constants depending only on κ and ε . Note that (28) is a direct consequence of Lemma 7 and (27) follows from (24) with $\theta = 1 - \alpha$. To prove (25) and (26), we introduce the difference terms

$$\begin{aligned} \gamma_0 &:= \frac{1}{u_1 - v_1} - \frac{1}{u_2 - v_2} = \frac{(u_2 - u_1) - (v_2 - v_1)}{(u_1 - v_1)(u_2 - v_2)}, \\ \gamma_1 &:= \frac{z'(u'_1 - v'_1) + v'_1}{u_1 - v_1} - \frac{z'(u'_2 - v'_2) + v'_2}{u_2 - v_2}, \\ \gamma_2 &:= \frac{1 + \varepsilon^2(z'(u'_1 - v'_1) + v'_1)^2}{(u_1 - v_1)^2} - \frac{1 + \varepsilon^2(z'(u'_2 - v'_2) + v'_2)^2}{(u_2 - v_2)^2}, \\ \gamma_3 &:= \frac{u'_1 - v'_1}{(u_1 - v_1)^2}(z'(u'_1 - v'_1) + v'_1) - \frac{u'_2 - v'_2}{(u_2 - v_2)^2}(z'(u'_2 - v'_2) + v'_2), \\ \gamma_4 &:= \frac{z'(u''_1 - v''_1) + v''_1}{u_1 - v_1} - \frac{z'(u''_2 - v''_2) + v''_2}{u_2 - v_2}, \end{aligned}$$

where u'_1 stands for $u_{1x'}$ et cetera. Consider $\Phi \in W_{2,D}^2(\Omega)$ and recall that $\mathcal{A}(u, v)\Phi \in L_2(\Omega) \hookrightarrow W_{2,D}^{-\alpha}(\Omega)$, so that for $\Psi \in W_{2,D}^\alpha(\Omega)$ we observe that

$$\begin{aligned} (29) \quad \int_{\Omega} [\mathcal{A}(u_1, v_1) - \mathcal{A}(u_2, v_2)] \Phi \Psi \, dx' dz' &= -2\varepsilon^2 \int_{\Omega} \gamma_1 \Phi_{x'z'} \Psi \, dx' dz' \\ &\quad + \int_{\Omega} \gamma_2 \Phi_{z'z'} \Psi \, dx' dz' + 2\varepsilon^2 \int_{\Omega} \gamma_3 \Phi_{z'} \Psi \, dx' dz' - \varepsilon^2 \int_{\Omega} \gamma_4 \Phi_{z'} \Psi \, dx' dz'. \end{aligned}$$

Rewriting γ_1 as

$$\gamma_1 = z' \left[\frac{u'_1 - u'_2}{u_1 - v_1} + u'_2 \gamma_0 \right] + (1 - z') \left[\frac{v'_1 - v'_2}{u_1 - v_1} + v'_2 \gamma_0 \right]$$

and using that $W_q^{2-\xi}(I) \hookrightarrow W_\infty^1(I)$, $W_{2,D}^\alpha(\Omega) \hookrightarrow L_2(\Omega)$ and that $(u_1, v_1), (u_2, v_2) \in \overline{S}_q(\kappa)$, one concludes that

$$\begin{aligned} (30) \quad \left| \int_{\Omega} \gamma_1 \Phi_{x'z'} \Psi \, dx' dz' \right| &\leq \|\gamma_1\|_{L_\infty(\Omega)} \|\Phi_{x'z'}\|_{L_2(\Omega)} \|\Psi\|_{L_2(\Omega)} \\ &\leq c_5(\kappa) \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)} \|\Phi\|_{W_{2,D}^2(\Omega)} \|\Psi\|_{W_{2,D}^\alpha(\Omega)}. \end{aligned}$$

Rewriting γ_2 and γ_3 as

$$\gamma_2 = \left(\frac{1}{u_1 - v_1} + \frac{1}{u_2 - v_2} \right) \gamma_0 + \varepsilon^2 \left(\frac{z'(u'_1 - v'_1) + v'_1}{u_1 - v_1} + \frac{z'(u'_2 - v'_2) + v'_2}{u_2 - v_2} \right) \gamma_1$$

and

$$\begin{aligned}\gamma_3 = & z' \left(\frac{u'_1 - v'_1}{u_1 - v_1} + \frac{u'_2 - v'_2}{u_2 - v_2} \right) \left(\frac{u'_1 - u'_2}{u_1 - v_1} + \frac{v'_2 - v'_1}{u_1 - v_1} + (u'_2 - v'_2)\gamma_0 \right) \\ & + \left(\frac{v'_2}{u_2 - v_2} + \frac{v'_1}{u_1 - v_1} \right) \left(\frac{v'_2 - v'_1}{u_2 - v_2} - v'_1\gamma_0 \right) + v'_1 \frac{u'_1 - u'_2}{(u_1 - v_1)^2} + u'_2 \frac{v'_1 - v'_2}{(u_2 - v_2)^2} \\ & + u'_2 v'_1 \left(\frac{1}{u_1 - v_1} + \frac{1}{u_2 - v_2} \right) \gamma_0\end{aligned}$$

it is clear that

$$\begin{aligned}(31) \quad & \left| \int_{\Omega} \gamma_2 \Phi_{z'} \Psi \, dx' dz' \right|, \left| \int_{\Omega} \gamma_3 \Phi_{z'} \Psi \, dx' dz' \right| \\ & \leq c_6(\kappa)(1 + \varepsilon^2) \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\varepsilon}(I) \times W_q^{2-\varepsilon}(I)} \|\Phi\|_{W_{2,D}^2(\Omega)} \|\Psi\|_{W_{2,D}^{\alpha}(\Omega)}.\end{aligned}$$

Writing γ_4 in the form

$$\gamma_4 = z' \left[\frac{u''_1 - u''_2}{u_1 - v_1} + u''_2 \gamma_0 \right] + (1 - z') \left[\frac{v''_1 - v''_2}{u_1 - v_1} + v''_2 \gamma_0 \right]$$

and applying the generalized Hölder inequality, the fourth integral in (29) can be estimated by

$$\begin{aligned}\left| \int_{\Omega} \gamma_4 \Phi_{z'} \Psi \, dx' dz' \right| & \leq \left| \int_{\Omega} \partial_{x'}^2(u_1 - u_2) \frac{z' \Phi_{z'} \Psi}{u_1 - v_1} \, dx' dz' \right| \\ & + \left| \int_{\Omega} \partial_{x'}^2(v_1 - v_2) \frac{(1 - z') \Phi_{z'} \Psi}{u_1 - v_1} \, dx' dz' \right| \\ & + \|\gamma_0\|_{L_{\infty}(I)} \left(\|u''_2\|_{L_q(I)} + \|v''_2\|_{L_q(I)} \right) \|\Phi_{z'}\|_{L_{2q/(q-2)}(\Omega)} \|\Psi\|_{L_2(\Omega)}.\end{aligned}$$

For $\xi \in [0, (q-1)/q]$ one has $(W_q^{\xi}(I))' = W_q^{-\xi}(I)$ so that

$$\left| \int_{\Omega} \partial_{x'}^2(u_1 - u_2) \frac{z' \Phi_{z'} \Psi}{u_1 - v_1} \, dx' dz' \right| \leq \|u_1 - u_2\|_{W_q^{2-\xi}(I)} \left\| \frac{1}{u_1 - v_1} \int_0^1 z' \Phi_{z'} \Psi \, dz' \right\|_{W_q^{\xi}(I)}.$$

As explained in the proof of [9, Lemma 2.4], the second factor is bounded by $\|\Phi_{z'}\|_{W_{2,D}^1(\Omega)} \|\Psi\|_{W_{2,D}^{\alpha}(\Omega)}$, up to a positive constant only depending on κ . Clearly, the same arguments apply to the integral involving the factor $\partial_{x'}^2(v_1 - v_2)$. Using that $W_2^1(\Omega) \hookrightarrow L_{2q/(q-2)}(\Omega)$ we infer

$$\begin{aligned}(32) \quad & \left| \int_{\Omega} \gamma_4 \Phi_{z'} \Psi \, dx' dz' \right| \\ & \leq c_7(\kappa) \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)} \|\Phi\|_{W_{2,D}^2(\Omega)} \|\Psi\|_{W_{2,D}^{\alpha}(\Omega)}.\end{aligned}$$

Now estimate (25) follows from (29)–(32). Analogously, one deduces from

$$\int_{\Omega} (f_{u_1, v_1} - f_{u_2, v_2}) \Psi \, dx' dz' = 2\varepsilon^2 \int_{\Omega} \gamma_3 \Psi \, dx' dz' - \varepsilon^2 \int_{\Omega} \gamma_4 \Psi \, dx' dz'$$

and the estimates

$$\left| \int_{\Omega} \gamma_3 \Psi \, dx' dz' \right| \leq c_8(\kappa) \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)} \|\Psi\|_{W_{2,D}^{\alpha}(\Omega)}$$

and

$$\begin{aligned}
\left| \int_{\Omega} \gamma_4 \Psi dx' dz' \right| &\leq \|u_1 - u_2\|_{W_q^{2-\xi}(I)} \left\| \frac{1}{u_1 - v_1} \int_0^1 z' \Psi dz' \right\|_{W_{q'}^{\xi}(I)} \\
&\quad + \|v_1 - v_2\|_{W_q^{2-\xi}(I)} \left\| \frac{1}{u_1 - v_1} \int_0^1 (1 - z') \Psi dz' \right\|_{W_{q'}^{\xi}(I)} \\
(33) \quad &\quad + \|\gamma_0\|_{L_{\infty}(I)} \left(\|u_2''\|_{L_2(I)} + \|v_2''\|_{L_2(I)} \right) \|\Psi\|_{L_2(\Omega)},
\end{aligned}$$

applying once more the technique of [9, Lemma 2.4] for the second factors of the first and second term on the right-hand side of (33), that (26) holds true. This completes the proof of the lemma. \square

A similar result with $\xi = 0$ and $\alpha = 0$ in the above lemma has been obtained in [25]. In the following lemma, we show that the transformed right-hand sides of (3)–(4) depend analytically and Lipschitz continuously on $(u, v) \in \overline{S}_q(\kappa)$. To simplify notation, we write u_x instead of $u_{x'}$ henceforth.

Lemma 9. *Let $q \in (2, \infty)$, $\kappa \in (0, 1/2)$, $\varepsilon > 0$, $2\sigma \in [0, 1/2)$ and $(u, v) \in \overline{S}_q(\kappa)$. Let $\tilde{\varphi}_{u,v;\varepsilon} \in W_2^{2\sigma}(\Omega)$ be the associated unique solution to (22)–(23). Then the mapping $g_{\varepsilon}: S_q(\kappa) \rightarrow W_{2,D}^{2\sigma}(I) \times W_{2,D}^{2\sigma}(I)$ defined by*

$$g_{\varepsilon}(u, v) = \left(\frac{1 + \varepsilon^2 u_x^2}{(u - v)^2} |\partial_{z'} \tilde{\varphi}_{u,v;\varepsilon}(\cdot, 1)|^2, \frac{1 + \varepsilon^2 v_x^2}{(u - v)^2} |\partial_{z'} \tilde{\varphi}_{u,v;\varepsilon}(\cdot, 0)|^2 \right)$$

is analytic, bounded, $g_{\varepsilon}(0, -1) = (1, 1)$, and if $\xi \in [0, 1/2)$ and $\nu \in [0, (1 - 2\xi)/2)$, then there exists a constant $C_4(\kappa, \varepsilon) > 0$ such that

$$(34) \quad \|g_{\varepsilon}(u_1, v_1) - g_{\varepsilon}(u_2, v_2)\|_{W_2^{\nu}(I) \times W_2^{\nu}(I)} \leq C_4 \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)}.$$

Proof. We first recall from Lemma 8 of [25] and the proof of Proposition 1 of [25] that, for any $(u, v) \in \overline{S}_q(\kappa)$,

$$\begin{aligned}
&\|\partial_{z'} \tilde{\varphi}_{u,v}(\cdot, 1)\|_{W_2^{1/2}(I)} + \|\partial_{z'} \tilde{\varphi}_{u,v}(\cdot, 0)\|_{W_2^{1/2}(I)} \\
&\quad + \| |\partial_{z'} \tilde{\varphi}_{u,v}(\cdot, 1)|^2 \|_{W_2^{2\sigma}(I)} + \| |\partial_{z'} \tilde{\varphi}_{u,v}(\cdot, 0)|^2 \|_{W_2^{2\sigma}(I)} \leq c_1(\kappa, \varepsilon).
\end{aligned}$$

We rewrite $g_{\varepsilon,1}(u_1, v_1) - g_{\varepsilon,1}(u_2, v_2)$ as the sum of the three terms

$$\begin{aligned}
I_1 &= \frac{1 + \varepsilon^2 (u_1')^2}{(u_1 - v_1)^2} (\partial_{z'} \tilde{\varphi}_{u_1, v_1}(\cdot, 1) + \partial_{z'} \tilde{\varphi}_{u_2, v_2}(\cdot, 1)) (\partial_{z'} \tilde{\varphi}_{u_1, v_1}(\cdot, 1) - \partial_{z'} \tilde{\varphi}_{u_2, v_2}(\cdot, 1)), \\
I_2 &= (1 + \varepsilon^2 (u_1')^2) |\partial_{z'} \tilde{\varphi}_{u_2, v_2}(\cdot, 1)|^2 \frac{u_2 - v_2 + u_1 - v_1}{(u_1 - v_1)^2 (u_2 - v_2)^2} (u_2 - u_1 + v_1 - v_2), \\
I_3 &= \varepsilon^2 |\partial_{z'} \tilde{\varphi}_{u_2, v_2}(\cdot, 1)|^2 \frac{u_1' + u_2'}{(u_2 - v_2)^2} (u_1' - u_2').
\end{aligned}$$

For $2\sigma \in (\xi + \nu, 1/2)$ and $s \in [\nu, 1 - \xi)$, $s \geq 1/q$, we have the continuous embeddings

$$W_q^s(I) \cdot W_2^{2\sigma}(I) \hookrightarrow W_2^{\nu}(I), \quad W_q^2(I) \cdot W_q^1(I) \cdot W_q^{1-\xi}(I) \hookrightarrow W_q^s(I)$$

and since $W_q^2(I)$ is an algebra, it is clear that

$$\|I_2\|_{W_2^{\nu}(I)}, \|I_3\|_{W_2^{\nu}(I)} \leq c_2(\kappa, \varepsilon) \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)}.$$

By $W_q^1(I) \cdot W_2^{1/2}(I) \cdot W_2^{1/2-\alpha}(I) \hookrightarrow W_2^\nu(I)$, the algebra property of $W_q^1(I)$ and the regularity properties of the trace operator, cf. [17, Theorem 1.5.1.1], we get

$$\begin{aligned} \|I_1\|_{W_2^\nu(I)} &\leq c_3(\kappa, \varepsilon) \|\tilde{\varphi}_{u_1, v_1} - \tilde{\varphi}_{u_2, v_2}\|_{W_{2,D}^{2-\alpha}(\Omega)} \\ &\leq c_4(\kappa, \varepsilon) \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)}, \end{aligned}$$

where we have used Lemma 8. The second component $g_{\varepsilon,2}(u_1, v_1) - g_{\varepsilon,2}(u_2, v_2)$ can be discussed similarly so that (34) follows. Analyticity of the map g_ε follows from the analyticity of the maps $\mathcal{A}^{-1}: S_q(\kappa) \rightarrow \mathcal{L}(L_2(\Omega), W_{2,D}^2(\Omega))$ and $[(u, v) \mapsto f_{u,v}], S_q(\kappa) \rightarrow L_2(\Omega)$. That $g_\varepsilon(0, -1) = (1, 1)$ and that g_ε is bounded is clear. \square

2.2. The abstract quasi-linear evolution equation. Let $q \in (2, \infty)$, $\xi \in (0, \frac{q-1}{q})$ and $\kappa \in (0, 1/2)$ and let $Z_q(\kappa)$ be the closed $1/\kappa$ -ball in $W_q^{2-\xi}(I)$. We define, for $w_1 \in Z_q(\kappa)$, the operator

$$(35) \quad A(w_1)w_2 := -\frac{w_{2xx}}{(1 + w_{1x}^2)^{3/2}}, \quad D(A(w_1)) = W_{q,D}^2(I).$$

Regarding (6), we introduce the function $\hat{v} = v + 1$ and $\hat{g}_\varepsilon(u, \hat{v}) = g_\varepsilon(u, \hat{v} - 1) = g_\varepsilon(u, v)$ to rewrite the problem (3)–(8) as

$$(36) \quad \frac{d}{dt} \begin{pmatrix} u \\ \hat{v} \end{pmatrix} + \begin{pmatrix} A(\varepsilon u) & 0 \\ 0 & A(\varepsilon \hat{v}) \end{pmatrix} \begin{pmatrix} u \\ \hat{v} \end{pmatrix} = \begin{pmatrix} -\lambda & 0 \\ 0 & \mu \end{pmatrix} \hat{g}_\varepsilon(u, \hat{v}), \quad t > 0,$$

$$(37) \quad \begin{pmatrix} u \\ \hat{v} \end{pmatrix} = \begin{pmatrix} u_0 \\ \hat{v}_0 \end{pmatrix}, \quad t = 0.$$

Note that the boundary conditions (5)–(6) are incorporated in the domain of the operator $A(\cdot)$. We now recall some important properties of $A(\cdot)$ from [9]: For $\omega > 0$ and $k \geq 1$ let $\mathcal{H}(W_{q,D}^2(I), L_q(I); k, \omega)$ be the set of all $A \in \mathcal{L}(W_{q,D}^2(I), L_q(I))$ such that $\omega + A$ is an isomorphism $W_{q,D}^2(I) \rightarrow L_q(I)$ satisfying

$$\frac{1}{k} \leq \frac{\|(\mu + A)z\|_{L_q(I)}}{\|\mu\| \|z\|_{L_q(I)} + \|z\|_{W_{q,D}^2(I)}} \leq k, \quad \operatorname{Re}(\mu) \geq \omega, \quad z \in W_{q,D}^2(I) \setminus \{0\}.$$

If $A \in \mathcal{H}(W_{q,D}^2(I), L_q(I); k, \omega)$, then $-A$ generates an analytic semigroup on $L_q(I)$ with domain $W_{q,D}^2(I)$. By [9, Lemma 3.1], for fixed $q \in (2, \infty)$, $\kappa \in (0, 1/2)$ and $\xi \in (0, (q-1)/q)$, there are $k(\kappa) \geq 1$ and $\omega(\kappa) > 0$ such that for any $w \in Z_q(\kappa)$, $-2\omega + A(w) \in \mathcal{H}(W_{q,D}^2(I), L_q(I); k, \omega)$ and $A(w)$ is resolvent positive satisfying

$$\|A(w_1) - A(w_2)\|_{\mathcal{L}(W_{q,D}^2(I), L_q(I))} \leq \ell(\kappa) \|w_1 - w_2\|_{W_q^{2-\xi}(I)}$$

with a positive constant $\ell(\kappa)$. For $\rho \in (0, 1)$ and $N, \tau > 0$ let

$$\begin{aligned} \mathcal{W}_\tau(\kappa) &:= \{w \in C([0, \tau], W_{q,D}^{2-\xi}(I)); \|w(t) - w(s)\|_{W_{q,D}^{2-\xi}(I)} \leq \frac{N}{\ell(\kappa)} |t - s|^\rho \\ &\text{and } w(t) \in Z_q(\kappa) \text{ for } 0 \leq t, s \leq \tau\}. \end{aligned}$$

By [9, Proposition 3.2], there is a constant $c_*(\rho) > 0$, independent of N, τ , such that for each $w \in \mathcal{W}_\tau(\kappa)$ there exists a unique parabolic evolution operator $\mathcal{U}_{A(w)}(t, s)$, $0 \leq s \leq t \leq \tau$, possessing $W_{q,D}^2(I)$ as a regular subspace and satisfying

$$\|\mathcal{U}_{A(w)}(t, s)\|_{\mathcal{L}(W_{q,D}^{2\alpha}(I), W_{q,D}^{2\beta}(I))} \leq c_{**}(\kappa)(t-s)^{\alpha-\beta} e^{-\vartheta(t-s)}, \quad 0 \leq s < t \leq \tau,$$

for $0 \leq \alpha \leq \beta \leq 1$ with $2\alpha, 2\beta \neq 1/q$. The constant $c_{**}(\kappa) \geq 1$ depends on N , α and β but is independent of τ and $-\vartheta = c_*(\rho)N^{1/\rho} - \omega(\kappa)$. Moreover $\mathcal{U}_{A(w)}(t, s) \in \mathcal{L}(L_q(I))$ is a positive operator for $0 \leq s \leq t \leq \tau$.

Let $\lambda, \mu > 0$, $q \in (2, \infty)$, $\varepsilon \in (0, 1)$ and fix $\kappa \in (0, 1/4)$. As in the proof of [9, Theorem 1.1], we also fix $0 < \xi < 1/q$, $0 < 1/2 - 1/q < 2\sigma < 1/2 - \xi$, $4\rho \in (0, \xi)$ and $N > 0$ such that $-\vartheta < 0$ and, for $w \in \mathcal{W}_\tau(\kappa)$ fixed,

$$(38) \quad \|\mathcal{U}_{A(w)}(t, s)\|_{\mathcal{L}(W_{q,D}^2(I))} + (t-s)^{-\sigma+1+\frac{1}{2}(\frac{1}{2}-\frac{1}{q})} \|\mathcal{U}_{A(w)}(t, s)\|_{\mathcal{L}(W_{2,D}^{2\sigma}(I), W_{q,D}^2(I))} \leq M(\kappa)e^{-\vartheta(t-s)},$$

for any $0 \leq s \leq t \leq \tau$, with a constant $M(\kappa) \geq 1$ independent of ω and τ . Note that (38) generalizes formula (34) of [25]. We consider $(u_0, v_0) \in W_q^2(I) \times W_q^2(I)$ satisfying $(u_0, v_0)(\pm 1) = (0, -1)$, $-1 \leq v_0 < u_0 \leq 0$ on I and

$$\|u_0\|_{W_{q,D}^2(I)}, \|\hat{v}_0\|_{W_{q,D}^2(I)} < \min \left\{ \frac{1}{4} - \kappa, \frac{\kappa}{2M} \right\} =: r(\kappa).$$

In view of the continuous embedding $W_q^2(I) \hookrightarrow L_\infty(I)$ with embedding constant 2, cf. the proof of Theorem 1.1 of [9], this implies that $(u_0, v_0) \in S_q(2\kappa)$ and that $\|\mathcal{U}_{A(w)}(t, 0)\hat{v}_0\|_{L_\infty(I)} < \kappa$ on $[0, \tau]$. Increasing M if necessary, we can also assume that $\|u_0\|_{W_{q,D}^{2-\xi}(I)}, \|\hat{v}_0\|_{W_{q,D}^{2-\xi}(I)} \leq 1/\kappa$ without loss of generality. Let $\kappa_0 = \kappa/M \leq \kappa$ and define the spaces

$$X_\tau(\kappa) := \left\{ (u, \hat{v}) \in \mathcal{W}_\tau(\kappa) \times \mathcal{W}_\tau(\kappa); \|u(t)\|_{W_{q,D}^2(I)}, \|\hat{v}(t)\|_{W_{q,D}^2(I)} \leq \frac{1}{\kappa_0} \right. \\ \left. 1 + u(t) - \hat{v}(t) \geq 2\kappa, \forall t \in [0, \tau] \right\}.$$

Then $X_\tau(\kappa) \subset \overline{S}_q(\kappa_0) + \{(0, 1)\}$, $\varepsilon X_\tau(\kappa) \subset X_\tau(\kappa)$ and $X_\tau(\kappa)$ is a complete metric space with respect to the topology of $C([0, \tau]; W_{q,D}^{2-\xi}(I) \times W_{q,D}^{2-\xi}(I))$. We now define, for $t \in [0, \tau]$ and $(u, \hat{v}) \in X_\tau(\kappa)$,

$$(39) \quad F(u, \hat{v})(t) := \begin{pmatrix} \mathcal{U}_{A(\varepsilon u)}(t, 0) & 0 \\ 0 & \mathcal{U}_{A(\varepsilon \hat{v})}(t, 0) \end{pmatrix} \begin{pmatrix} u_0 \\ \hat{v}_0 \end{pmatrix} \\ + \int_0^t \begin{pmatrix} -\lambda \mathcal{U}_{A(\varepsilon u)}(t, s) & 0 \\ 0 & \mu \mathcal{U}_{A(\varepsilon \hat{v})}(t, s) \end{pmatrix} \hat{g}_\varepsilon(u(s), \hat{v}(s)) ds$$

and claim that $F: X_\tau(\kappa) \rightarrow X_\tau(\kappa)$ is a contraction for either (λ, μ) arbitrary and τ sufficiently small or for (λ, μ) and (u_0, \hat{v}_0) small and τ arbitrary. Recall from Lemma 9 that, for $(u_1, v_1), (u_2, v_2) \in \overline{S}_q(\kappa)$,

$$\|\hat{g}_\varepsilon(u_1, \hat{v}_1) - \hat{g}_\varepsilon(u_2, \hat{v}_2)\|_{W_{2,D}^{2\sigma}(I) \times W_{2,D}^{2\sigma}(I)} \\ \leq C_4(\kappa, \varepsilon) \|(u_1, v_1) - (u_2, v_2)\|_{W_q^{2-\xi}(I) \times W_q^{2-\xi}(I)},$$

and that

$$(40) \quad \|\hat{g}_\varepsilon(u, \hat{v})\|_{W_{2,D}^{2\sigma}(I) \times W_{2,D}^{2\sigma}(I)} \leq C_5(\kappa, \varepsilon), \quad \forall (u, v) \in \overline{S}_q(\kappa).$$

Let

$$\mathcal{I}(\tau) := \int_0^\tau e^{-\vartheta s} s^{\sigma-1-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})} ds.$$

Then $\mathcal{I} \rightarrow 0$ as $\tau \rightarrow 0$, $\mathcal{I} \rightarrow \mathcal{I}(\infty) < \infty$ for $\tau \rightarrow \infty$ and $\tau \mapsto \mathcal{I}(\tau)$ is monotonically increasing on $[0, \infty)$. Using that $W_q^2(I) \hookrightarrow L_\infty(I)$ with embedding constant 2 together with the positivity of the evolution operator and (40), one concludes from Eq. (39) that, for $i = 1, 2$,

$$(41) \quad 0 \geq F_1(u, \hat{v})(t),$$

$$(42) \quad 0 \leq F_2(u, \hat{v})(t),$$

$$(43)$$

$$1 + F_1(u, \hat{v})(t) - F_2(u, \hat{v})(t) \geq 4\kappa - \mathcal{U}_{A(\varepsilon\hat{v})}(t, 0)\hat{v}_0 - 2(\lambda + \mu)M(\kappa)C_5(\kappa, \varepsilon)\mathcal{I}(\tau) \text{ and}$$

$$(44) \quad \|F_i(u, \hat{v})(t)\|_{W_{q,D}^2(I)} \leq \frac{1}{2\kappa_0} + \max\{\lambda, \mu\}M(\kappa)C_5(\kappa, \varepsilon)\mathcal{I}(\tau).$$

Applying [3, II. Theorem 5.2.1] with $\alpha = 1$, $\beta = 1 - \xi/2$ and $2\gamma = 2\sigma - 1/2 + 1/q$ together with

$$W_{2,D}^{2\sigma}(I) \hookrightarrow W_{q,D}^{2\sigma - \frac{1}{2} + \frac{1}{q}}(I) \hookrightarrow L_q(I)$$

we see that there exists a constant $C_6(\kappa) > 0$ such that, with $m = \max\{\lambda, \mu\}$,

$$(45)$$

$$\begin{aligned} & \|F(u_1, \hat{v}_1)(t) - F(u_2, \hat{v}_2)(t)\|_{W_{q,D}^{2-\xi}(I) \times W_{q,D}^{2-\xi}(I)} \leq C_6 \left(m \max_{0 \leq t \leq \tau} \left(t^{\frac{\xi}{2} + \sigma - \frac{1}{2}(\frac{1}{2} - \frac{1}{q})} e^{-\vartheta t} \right) \right. \\ & \left. + \left(m + \frac{1}{2} \|(u_0, \hat{v}_0)\|_{W_{q,D}^2(I) \times W_{q,D}^2(I)} \right) \max_{0 \leq t \leq \tau} \left(t^{\frac{\xi}{2}} e^{-\vartheta t} \right) \right) \|(u_1, \hat{v}_1) - (u_2, \hat{v}_2)\|_{X_\tau(\kappa)}. \end{aligned}$$

Applying [3, II. Theorem 5.3.1] with $2\alpha = 2 - \xi + 4\rho$ and $2\beta = 2 - \xi$ together with the embedding

$$W_{q,D}^2(I) \hookrightarrow W_{q,D}^{2-\xi+4\rho}(I)$$

we obtain, for $i = 1, 2$, $0 \leq s \leq t \leq \tau$ and $(u, \hat{v}) \in X_\tau(\kappa)$,

$$(46) \quad \begin{aligned} & \|F_i(u, \hat{v})(t) - F_i(u, \hat{v})(s)\|_{W_{q,D}^{2-\xi}(I)} \leq C_7 \max_{0 \leq t \leq \tau} (t^\rho e^{-\vartheta t}) \\ & \times \left(\|(u_0, \hat{v}_0)\|_{W_{q,D}^{2-\xi+4\rho}(I) \times W_{q,D}^{2-\xi+4\rho}(I)} + 2mC_5 \right) (t - s)^\rho \end{aligned}$$

where $C_7(\kappa) > 0$. As $F(u, \hat{v})(0) = (u_0, \hat{v}_0)$, we conclude from (46) and the triangle inequality that

$$(47) \quad \begin{aligned} & \|F_i(u, \hat{v})(t)\|_{W_{q,D}^{2-\xi}(I)} \leq C_7 \max_{0 \leq t \leq \tau} (t^{2\rho} e^{-\vartheta t}) \\ & \times \left(\|(u_0, \hat{v}_0)\|_{W_{q,D}^{2-\xi+4\rho}(I) \times W_{q,D}^{2-\xi+4\rho}(I)} + 2mC_5 \right) + \frac{1}{2\kappa}. \end{aligned}$$

It follows from (43)–(47) that we can choose $\tau > 0$ sufficiently small so that $F: X_\tau(\kappa) \rightarrow X_\tau(\kappa)$ is indeed a contraction. The unique fixed point of F in $X_\tau(\kappa)$ is a mild solution to (36)–(37) which can, according to [1, Theorem 4.2] and [2, Theorem 10.1], be extended to a strong solution on a maximal interval of existence with the regularity specified in Theorem 1. Regarding (41), (42), Theorem 1 follows from arguments very similar to what is presented in the proof of [25, Theorem 2] and [9, Theorem 1.1].

2.3. The small aspect ratio limit. We now establish that there is a positive ε -independent lower bound for the maximal existence times T_ε of solutions $(u_\varepsilon, v_\varepsilon, \varphi_\varepsilon)$ to (1)–(8) as $\varepsilon \rightarrow 0$. Then Theorem 2 follows from arguments very similar to what is presented in the proof of [25, Theorem 10] and [9, Theorem 1.4].

Fix $\lambda, \mu > 0$, $q \in (2, \infty)$ and $\kappa \in (0, 1/2)$ and consider $(u_0, v_0) \in W_q^2(I) \times W_q^2(I)$ with $(u_0, v_0)(\pm 1) = (0, -1)$, $-1 \leq v_0 < u_0 \leq 0$ and $\|u_0\|_{W_{q,D}^2(I)}, \|v_0 + 1\|_{W_{q,D}^2(I)} < r(\kappa/2)$ so that $(u_0, v_0) \in S_q(\kappa)$. For $\varepsilon \in (0, 1)$ we denote by $(u_\varepsilon, v_\varepsilon, \varphi_\varepsilon)$ the unique solution to (1)–(8) with initial values (u_0, v_0) , defined on the maximal interval $[0, T_\varepsilon)$. Let $\kappa_1 := \kappa/(2M) < \kappa$ with M as in (38) and

$$\tau_\varepsilon := \sup \{t \in [0, T_\varepsilon); (u_\varepsilon(s), v_\varepsilon(s)) \in \overline{S}_q(\kappa_1); \forall s \in [0, t]\} > 0.$$

We then have $T_\varepsilon \geq \tau_\varepsilon$,

$$u_\varepsilon(t) - v_\varepsilon(t) \geq 2\kappa_1, \quad -1 \leq v_\varepsilon(t) < u_\varepsilon(t) \leq 0 \quad \text{on } [0, \tau_\varepsilon] \times [-1, 1],$$

and, by the continuous embedding $W_q^2(I) \hookrightarrow W_\infty^1(I)$,

$$\|u_\varepsilon(t)\|_{W_q^2(I)} + \|v_\varepsilon(t)\|_{W_q^2(I)} + \|u_\varepsilon(t)\|_{W_\infty^1(I)} + \|v_\varepsilon(t)\|_{W_\infty^1(I)} \leq C_8(\kappa), \quad \forall t \in [0, \tau_\varepsilon].$$

Henceforth, we choose ε sufficiently small, precisely, ε smaller than some $\varepsilon_* \in (0, 1)$, so that

$$\varepsilon_*^2 \left(\|u_{\varepsilon,x}(t)\|_{L_\infty(I)} + 2 \|v_{\varepsilon,x}(t)\|_{L_\infty(I)} \right)^2 \leq \frac{1}{2}, \quad \forall (t, \varepsilon) \in [0, \tau_\varepsilon] \times (0, \varepsilon_*].$$

For $(t, x', z') \in [0, \tau_\varepsilon] \times \overline{\Omega}$, we recall the definition $\psi_\varepsilon(t, x', z') = \tilde{\varphi}_\varepsilon(t, x', z') - z'$, where $\tilde{\varphi}_\varepsilon(t, x', z') = \theta_*(u(t), v(t))\varphi_\varepsilon$. Then $\psi_\varepsilon(t)$ satisfies the uniform estimates established in [25, Lemma 8]. The fact that multiplication $W_q^1(I) \cdot W_2^{1/2}(I) \cdot W_2^{1/2}(I) \hookrightarrow W_2^{2\sigma}(I)$, $2\sigma \in (0, 1/2)$, is continuous implies that

$$(48) \quad \|g_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))\|_{W_2^{2\sigma}(I) \times W_2^{2\sigma}(I)} \leq C_9(\kappa).$$

Using (38), (39), (48) and that $(u_0, v_0) \in S_q(\kappa)$ we get

$$(49) \quad \|u_\varepsilon(t)\|_{W_{q,D}^2(I)}, \|v_\varepsilon(t) + 1\|_{W_{q,D}^2(I)} \leq \frac{M}{\kappa} + mMC_9\mathcal{I}(t),$$

with $m = \max\{\lambda, \mu\}$. Regarding (41)–(43), we recall that

$$(50) \quad u_\varepsilon(t) \leq 0,$$

$$(51) \quad v_\varepsilon(t) \geq -1 \text{ and}$$

$$(52) \quad u_\varepsilon(t) - v_\varepsilon(t) \geq 2\kappa - \mathcal{U}_{A(\varepsilon(v+1))}(t, 0)(v_0 + 1) - 2(\lambda + \mu)MC_9\mathcal{I}(t).$$

As $\|v_0 + 1\|_{W_{q,D}^2(I)} < \frac{2M-1}{4M^2}\kappa$, we have that $\|\mathcal{U}_{A(\varepsilon(v+1))}(t, 0)(v_0 + 1)\|_{L_\infty(I)} < \kappa - \kappa_1$. Furthermore, there exists $\tau > 0$ such that $2(\lambda + \mu)MC_9\mathcal{I}(t) \leq \kappa - \kappa_1$ on $[0, \tau]$. Decreasing τ if necessary to guarantee that $mMC_9\mathcal{I}(t) \leq M/\kappa$, we conclude from (49) and (52) that $(u_\varepsilon, v_\varepsilon)(t) \in \overline{S}_q(\kappa_1)$ for all $t \in [0, \tau] \cap [0, \tau_\varepsilon]$ and in particular $\tau_\varepsilon \geq \tau$. With

$$\Lambda(\kappa) := \min \left\{ \frac{1}{\kappa C_9\mathcal{I}(\infty)}, \frac{(2M-1)\kappa}{8M^2 C_9\mathcal{I}(\infty)} \right\}$$

it is also clear that, for $\lambda, \mu < \Lambda(\kappa)$, we obtain from (49) and (52) that $\tau_\varepsilon \geq \tau$ for any $\tau > 0$ and this implies that $T_\varepsilon = \infty$.

3. THE NON-EXISTENCE OF GLOBAL SOLUTIONS

In this chapter, we first focus on the stationary version of (1)–(8), i.e., the problem

$$(53) \quad -\Delta_\varepsilon \varphi = 0, \quad \text{in } \Omega_{u,v},$$

$$(54) \quad \varphi = \frac{z-v}{u-v}, \quad \text{on } \partial\Omega_{u,v},$$

$$(55) \quad u_{xx} = \lambda(1 + \varepsilon^2 u_x^2)^{5/2} |\varphi_z(x, u(x))|^2, \quad x \in I,$$

$$(56) \quad v_{xx} = -\mu(1 + \varepsilon^2 v_x^2)^{5/2} |\varphi_z(x, v(x))|^2, \quad x \in I,$$

$$(57) \quad u(\pm 1) = 0,$$

$$(58) \quad v(\pm 1) = -1.$$

Let us introduce the functions

$$J(r) := \int_0^r \frac{ds}{(1+s^2)^{5/2}} = \frac{r(2r^2+3)}{3(r^2+1)^{3/2}}, \quad \tilde{J}(r) := J(-r)$$

and

$$\xi_0(\varepsilon) := \min \left\{ \frac{2J(\varepsilon)}{\varepsilon}, \frac{2}{3\varepsilon} \right\}.$$

Note that J is strictly increasing, concave and maps $[0, \infty)$ to $[0, 2/3]$. It has been shown in the proof of [25, Theorem 5] that the potential satisfies

$$(59) \quad z - v(x) \leq \varphi(x, z) \leq 1 + z - u(x), \quad \forall (x, z) \in \overline{\Omega}_{u,v}.$$

Using the upper bound in (59) and the function J , the methods used in the proof of [9, Theorem 1.3] imply that there can be no solution of (55) with boundary condition (57) provided $\lambda > \xi_0(\varepsilon)$. We now make use of the lower bound for φ and infer from $\varphi_z(x, v(x)) \geq 1$ and (56) that

$$\frac{v_{xx}}{(1 + \varepsilon^2 v_x^2)^{5/2}} = -\frac{1}{\varepsilon} \partial_x \tilde{J}(\varepsilon v_x) \leq -\mu.$$

Without loss of generality, we assume that v attains a maximum at $x_m \in (-1, 0]$. Integrating the above inequality over $[x_m, x]$ for $x \in [0, 1]$ implies that

$$\tilde{J}(\varepsilon v_x) \geq \mu \varepsilon x, \quad x \in [0, 1].$$

Now either $\mu \varepsilon \geq 2/3$ and then $\tilde{J}(\varepsilon v_x(1)) \geq 2/3$ which implies that $v_x(1) = -\infty$, a contradiction, or $\mu \varepsilon < 2/3$ and then, by Jensen's inequality,

$$\tilde{J}(-\varepsilon v(0) - \varepsilon) = \tilde{J} \left(\int_0^1 \varepsilon v_x dx \right) \geq \int_0^1 \tilde{J}(\varepsilon v_x) dx \geq \mu \frac{\varepsilon}{2}.$$

If $\mu > 2J(\varepsilon)/\varepsilon$, we obtain that $J(\varepsilon v(0) + \varepsilon) > J(\varepsilon)$, i.e., $v(0) > 0$, which is again a contradiction. This completes the proof of Theorem 3.

We now present a proof of Theorem 4 and begin with the following lemma which refines the estimates (59). Recall that, by our assumptions, we concentrate on solutions (u, v) to (1)–(8) such that $v, -u - 1 \leq c$, for some $c < 0$, and that (u, v) stays bounded in $W_q^2(I) \times W_q^2(I)$, i.e., by Sobolev's embedding theorem, $\|u\|_{C^1([-1,1])}$ and $\|v\|_{C^1([-1,1])}$ are bounded by a positive constant only depending on q .

Lemma 10. *Let $\varphi \in W_2^2(\Omega_{u,v})$ be a solution of $-\Delta_\varepsilon \varphi = 0$ satisfying the boundary conditions $\varphi(\pm 1, z) = 1 + z$, $\varphi(x, u(x)) = 1$ and $\varphi(x, v(x)) = 0$. Then there is $n \in 2\mathbb{N}$ such that, for all $(x, z) \in \overline{\Omega}_{u,v}$,*

$$x^n + z \leq \varphi(x, z) \leq 2 + z - x^n.$$

Proof. For some $n \in 2\mathbb{N}$, let $S_n^-(x, z) = x^n + z$ and $S_n^+(x, z) = 2 + z - x^n$. Then

$$-\Delta_\varepsilon S_n^- = (-\varepsilon^2 \partial_x^2 - \partial_z^2) S_n^- = -\varepsilon^2 n(n-1) x^{n-2} \leq 0$$

and we observe that

$$S_n^-(\pm 1, z) = 1 + z = \varphi(\pm 1, z)$$

and

$$S_n^-(x, u(x)) = x^n + u(x) \leq 1 = \varphi(x, u(x)).$$

As v_x is uniformly bounded by a constant only depending on q and by $v \leq c$, $c < 0$, we shall make use of the fact that $x^n \rightarrow 0$, $n \rightarrow \infty$, pointwise in I , to obtain that $v(x) \leq -x^n$ or equivalently

$$S_n^-(x, v(x)) \leq \varphi(x, v(x)),$$

for some $n \in 2\mathbb{N}$ and all $x \in I$. As

$$-\Delta_\varepsilon (S_n^- - \varphi) \leq 0 \text{ in } \Omega \quad \text{and} \quad (S_n^- - \varphi)|_{\partial\Omega} \leq 0,$$

we can apply the weak maximum principle to conclude that $S_n^- \leq \varphi$ in $\overline{\Omega}_{u,v}$. Similarly, one shows that

$$-\Delta_\varepsilon (S_n^+ - \varphi) \geq 0 \text{ in } \Omega \quad \text{and} \quad (S_n^+ - \varphi)|_{\partial\Omega} \geq 0$$

so that the weak maximum principle implies that $S_n^+ \geq \varphi$ in $\overline{\Omega}_{u,v}$. \square

Note that the number n in the above lemma only depends on c and q . Let us now modify the calculations in [10] for the problem under discussion.

We multiply $\varepsilon^2 \varphi_{xx} + \varphi_{zz} = 0$ by the function $\varphi_z - 1$, integrate over $\Omega_{u,v}$ and use integration by parts to obtain that

$$\begin{aligned} 0 &= -\varepsilon^2 \int_{\Omega_{u,v}} \varphi_x \varphi_{xz} dx dz + \varepsilon^2 \int_{\partial\Omega_{u,v}} \varphi_x (\varphi_z - 1) n_1 ds \\ &\quad + \int_{\Omega_{u,v}} \left(\frac{1}{2} \frac{d}{dz} \varphi_z^2 - \varphi_{zz} \right) dx dz, \end{aligned}$$

with $n = (n_1, n_2)$ denoting the outward normal of $\partial\Omega_{u,v}$. Using the identities

$$(60) \quad \varphi_x(x, u(x)) = -u_x \varphi_z(x, u(x)),$$

$$(61) \quad \varphi_x(x, v(x)) = -v_x \varphi_z(x, v(x)),$$

which follow from differentiating the boundary conditions $\varphi(x, u(x)) = 1$ and $\varphi(x, v(x)) = 0$, and $\varphi_z(\pm 1, z) = 1$ we obtain

$$\begin{aligned}
0 &= -\frac{\varepsilon^2}{2} \int_{\Omega_{u,v}} \frac{d}{dz} \varphi_x^2 dx dz + \varepsilon^2 \int_I \varphi_z(x, u(x)) (\varphi_z(x, u(x)) - 1) u_x^2 dx \\
&\quad - \varepsilon^2 \int_I \varphi_z(x, v(x)) (\varphi_z(x, v(x)) - 1) v_x^2 dx + \frac{1}{2} \int_I (\varphi_z^2(x, u(x)) - \varphi_z^2(x, v(x))) dx \\
&\quad - \int_I (\varphi_z(x, u(x)) - \varphi_z(x, v(x))) dx \\
&= \frac{\varepsilon^2}{2} \int_I (\varphi_z^2(x, u(x)) u_x^2 - \varphi_z^2(x, v(x)) v_x^2) dx + \frac{1}{2} \int_I (\varphi_z^2(x, u(x)) - \varphi_z^2(x, v(x))) dx \\
&\quad - \varepsilon^2 \int_I (\varphi_z(x, u(x)) u_x^2 - \varphi_z(x, v(x)) v_x^2) dx - \int_I (\varphi_z(x, u(x)) - \varphi_z(x, v(x))) dx
\end{aligned}$$

and thus

$$\begin{aligned}
\int_I (1 + \varepsilon^2 v_x^2) \varphi_z^2(x, v(x)) dx &= \int_I (1 + \varepsilon^2 u_x^2) (\varphi_z^2 - 2\varphi_z)(x, u(x)) dx \\
&\quad + 2 \int_I (1 + \varepsilon^2 v_x^2) \varphi_z(x, v(x)) dx \\
&\geq 2 \int_I (1 + \varepsilon^2 v_x^2) \varphi_z(x, v(x)) dx - \int_I (1 + \varepsilon^2 u_x^2) dx.
\end{aligned}$$

As $\|u\|_{C^1([-1,1])}$ is bounded by a constant only depending on q , there is $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, we have that $\varepsilon^2 \|u_x\|_{L^\infty(I)}^2 \leq \frac{1}{4}$ and thus

$$(62) \quad \int_I (1 + \varepsilon^2 v_x^2) \varphi_z^2(x, v(x)) dx \geq 2 \int_I (1 + \varepsilon^2 v_x^2) \varphi_z(x, v(x)) dx - \frac{5}{2}.$$

A corresponding estimate with v replaced by u can be obtained similarly. Working with (62) henceforth motivates to assume that $\mu \geq \lambda$ in the following, without loss of generality.

We multiply $\varepsilon^2 \varphi_{xx} + \varphi_{zz} = 0$ by the function $\varphi - 1$, integrate over $\Omega_{u,v}$ and use integration by parts, (61) and Theorem 1.(iii) to obtain that

$$\begin{aligned}
\int_{\Omega_{u,v}} (\varepsilon^2 \varphi_x^2 + \varphi_z^2) dx dz &= \varepsilon^2 \int_{\partial\Omega_{u,v}} \varphi_x(\varphi - 1) n_1 ds + \int_{\partial\Omega_{u,v}} \varphi_z(\varphi - 1) n_2 ds \\
&= -\varepsilon^2 \int_I \varphi_x(x, v(x)) v_x dx + \varepsilon^2 \int_{-1}^0 \varphi_x(1, z) z dz \\
&\quad - \varepsilon^2 \int_{-1}^0 \varphi_x(-1, z) z dz + \int_I \varphi_z(x, v(x)) dx \\
&= \int_I (1 + \varepsilon^2 v_x^2) \varphi_z(x, v(x)) dx + 2\varepsilon^2 \int_{-1}^0 \varphi_x(1, z) z dz.
\end{aligned}$$

By Lemma 10,

$$\varphi(x, z) - \varphi(1, z) \leq 2 + z - x^n - (1 + z) = -(x^n - 1)$$

and, for $x < 1$, we obtain

$$\frac{\varphi(x, z) - \varphi(1, z)}{x - 1} \geq -x^{n-1} - x^{n-2} - \dots - x - 1.$$

Sending $x \rightarrow 1$, we conclude that $\varphi_x(1, z) \geq -n$ and thus, as $z \in [-1, 0]$,

$$\int_{-1}^0 \varphi_x(1, z) z \, dz \leq \frac{n}{2}.$$

This yields

$$(63) \quad \int_{\Omega_{u,v}} (\varepsilon^2 \varphi_x^2 + \varphi_z^2) \, dx \, dz \leq \int_I (1 + \varepsilon^2 v_x^2) \varphi_z(x, v(x)) \, dx + \varepsilon^2 n.$$

Now

$$(64) \quad \frac{1}{u-v} = \frac{(\varphi(x, u(x)) - \varphi(x, v(x)))^2}{u-v} = \frac{1}{u-v} \left(\int_{v(x)}^{u(x)} \varphi_z(x, z) \, dz \right)^2 \\ \leq \int_{\Omega_{u,v}} (\varepsilon^2 \varphi_x^2 + \varphi_z^2) \, dx \, dz.$$

The function $\alpha(r) := \frac{1}{1+r}$, $r \in (-1, \infty)$ is convex and Jensen's inequality implies

$$\frac{1}{2} \int_I \frac{1}{u-v} \, dx = \frac{1}{2} \int_I \frac{1}{1 + [u - (v+1)]} \, dx \geq \frac{1}{1 + \frac{1}{2} \int_I [u - (v+1)] \, dx}.$$

Setting

$$E(t) := -\frac{1}{2} \int_I [u - (v+1)] \, dx,$$

we note that $E(t) \in [0, 1)$, and using (62), (63) and (64), we derive the inequality

$$\frac{1}{1-E(t)} \leq \frac{1}{2} \int_{\Omega_{u,v}} (\varepsilon^2 \varphi_x^2 + \varphi_z^2) \, dx \, dz \\ \leq \frac{1}{2} \int_I (1 + \varepsilon^2 v_x^2) \varphi_z(x, v(x)) \, dx + \varepsilon^2 \frac{n}{2} \\ \leq \frac{1}{4} \left(\int_I (1 + \varepsilon^2 v_x^2) \varphi_z^2(x, v(x)) \, dx + \frac{5}{2} \right) + \varepsilon^2 \frac{n}{2}.$$

In view of (3) and (4), we observe that

$$\frac{dE(t)}{dt} = - \left[\frac{u_x}{2\sqrt{1+\varepsilon^2 u_x^2}} - \frac{v_x}{2\sqrt{1+\varepsilon^2 v_x^2}} \right]_{-1}^1 + \frac{\lambda}{2} \int_I \varphi_z^2(x, u(x)) (1 + \varepsilon^2 u_x^2) \, dx \\ + \frac{\mu}{2} \int_I \varphi_z^2(x, v(x)) (1 + \varepsilon^2 v_x^2) \, dx \\ \geq -\frac{2}{\varepsilon} + \frac{\mu}{2} \left(\frac{4}{1-E(t)} - \frac{5}{2} - 2\varepsilon^2 n \right).$$

If necessary, we decrease $\varepsilon_0 > 0$ to guarantee that $2\varepsilon^2 n \leq \frac{1}{2}$ and so

$$\frac{dE(t)}{dt} \geq -\frac{2}{\varepsilon} + 2\mu\alpha(-E) - \frac{3}{2}\mu =: F_\mu(E).$$

As $-1 < -E \leq 0$, $1 \leq \alpha(-E) < \infty$ and hence $F_\mu(E) \geq -2/\varepsilon + \mu/2$. If $\mu > 4/\varepsilon$, then $F_\mu(E) > 0$ and the above inequality implies that $E(t)$ is strictly increasing. As $F_\mu(E)$ is also strictly increasing, we must have

$$\frac{dE(t)}{dt} \geq F_\mu(E(0)) \geq F_\mu(0).$$

This shows that

$$1 > E(t) \geq E(0) + F_\mu(0)t, \quad \forall t \in [0, T_\varepsilon),$$

which immediately yields

$$T_\varepsilon \leq T_\varepsilon^* := \frac{1 - E(0)}{F_\mu(0)} = \frac{1}{\mu - 4/\varepsilon} \int_I (u_0 - v_0) dx$$

and hence (17). Moreover, $0 \leq \min_{x \in I} \{u(t) - v(t)\} \leq 1 - E(t)$ so that, for $T_\varepsilon = T_\varepsilon^*$, $\liminf_{t \rightarrow T_\varepsilon} \min_{x \in I} \{u(t) - v(t)\} = 0$. This completes the proof of Theorem 4.

4. ASYMPTOTICALLY STABLE STEADY STATE SOLUTIONS

In terms of the coordinates $(x', z') \in \overline{\Omega}$, the problem (53)–(58) reads

$$(65) \quad -\tilde{\Delta}_\varepsilon \tilde{\varphi} = 0, \quad \text{in } \Omega,$$

$$(66) \quad \tilde{\varphi} = z', \quad \text{on } \partial\Omega,$$

$$(67) \quad u_{x'x'} = \lambda \frac{(1 + \varepsilon^2 u_{x'}^2)^{5/2}}{(u - v)^2} |\tilde{\varphi}_{z'}(x', 1)|^2, \quad x' \in I,$$

$$(68) \quad v_{x'x'} = -\mu \frac{(1 + \varepsilon^2 v_{x'}^2)^{5/2}}{(u - v)^2} |\tilde{\varphi}_{z'}(x', 0)|^2, \quad x' \in I,$$

$$(69) \quad u(\pm 1) = 0,$$

$$(70) \quad v(\pm 1) = -1.$$

Fix $q \in (2, \infty)$ and $\kappa \in (0, 1/2)$. We recall the notation $\hat{v} = v + 1$ and the definition of the operator A in (35). Defining $h_\varepsilon := (h_{1,\varepsilon}, h_{2,\varepsilon}): S_q(\kappa) \rightarrow L_q(I) \times L_q(I)$ by

$$h_{1,\varepsilon}(u, v) := \frac{(1 + \varepsilon^2 u_{x'}^2)^{5/2}}{(u - v)^2} |\tilde{\varphi}_{z'}(x', 1)|^2$$

$$h_{2,\varepsilon}(u, v) := \frac{(1 + \varepsilon^2 v_{x'}^2)^{5/2}}{(u - v)^2} |\tilde{\varphi}_{z'}(x', 0)|^2$$

and recalling that $-A(0) = \partial_{x'}^2 \in \mathcal{L}(W_{q,D}^2(I), L_q(I))$ is invertible, we introduce a map $F: \mathbb{R}^2 \times S_q(\kappa) \rightarrow W_{q,D}^2(I) \times W_q^2(I)$ by

$$F(\Lambda, U) := \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + \begin{pmatrix} \Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix} A(0)^{-1} h_\varepsilon(U_1, U_2).$$

Then $F(0, 0) = (0, 0)$ and $D_U F(0, 0) = \text{id}$ so that, in view of the Implicit Function Theorem, there is $\delta = \delta(\kappa) > 0$ and an analytic map $[\Lambda \mapsto U_\Lambda]: [0, \delta]^2 \rightarrow W_{q,D}^2(I) \times W_q^2(I)$ satisfying $F(\Lambda, U_\Lambda) = 0$. For $\Lambda \neq (0, 0)$, let Φ_Λ be the potential associated

with U_Λ . Then $(U_\Lambda, \Phi_\Lambda) \in S_q(\kappa) \times W_2^2(\Omega)$ is the unique stationary solution to (1)–(8). Given $U = (U_1, U_2)$, we use the notation $\hat{U} = (U_1, \hat{U}_2)$ and we write $\Lambda = (\lambda, \mu)$. Letting $\hat{V} = U - U_\Lambda = \hat{U} - \hat{U}_\Lambda$ and introducing a map $Q = (Q_1, Q_2)$ by setting

$$\begin{aligned} Q_1(u, \hat{v}) &:= -A(\varepsilon u)u - \lambda \hat{g}_{\varepsilon,1}(u, \hat{v}), \\ Q_2(u, \hat{v}) &:= -A(\varepsilon \hat{v})\hat{v} + \mu \hat{g}_{\varepsilon,2}(u, \hat{v}), \end{aligned}$$

we observe that $Q(\hat{U}_\Lambda) = 0$, and we introduce the function

$$G_\Lambda(\hat{V}) := Q(\hat{V} + \hat{U}_\Lambda) - DQ(\hat{U}_\Lambda)\hat{V}$$

so that, for \hat{U} being a solution of (3)–(4),

$$\frac{d}{dt}\hat{V} - DQ(\hat{U}_\Lambda)\hat{V} = G_\Lambda(\hat{V}).$$

Clearly, $G_\Lambda \in C^\infty(\mathcal{O}_\Lambda, L_q(I) \times L_q(I))$, where $\mathcal{O}_\Lambda \subset W_{q,D}^2(I) \times W_{q,D}^2(I)$ is a neighborhood of zero such that $U_\Lambda + \mathcal{O}_\Lambda \subset S_q(\kappa)$, $G_\Lambda(0) = 0$ and $DG_\Lambda(0) = 0$. A straightforward computation shows that

$$\begin{aligned} -DQ(\hat{U}_\Lambda)\hat{V} &= \begin{pmatrix} A(\varepsilon U_{\Lambda,1}) & 0 \\ 0 & A(\varepsilon \hat{U}_{\Lambda,2}) \end{pmatrix} \hat{V} \\ &\quad + 3\varepsilon^2 \begin{pmatrix} \lambda g_{\varepsilon,1}(U_\Lambda) \frac{\partial_{x'} U_{\Lambda,1}}{1+\varepsilon^2(\partial_{x'} U_{\Lambda,1})^2} & 0 \\ 0 & -\mu g_{\varepsilon,2}(U_\Lambda) \frac{\partial_{x'} U_{\Lambda,2}}{1+\varepsilon^2(\partial_{x'} U_{\Lambda,2})^2} \end{pmatrix} \partial_{x'} \hat{V} \\ &\quad + \begin{pmatrix} \lambda & 0 \\ 0 & -\mu \end{pmatrix} D\hat{g}_\varepsilon(\hat{U}_\Lambda)\hat{V} \\ &=: \begin{pmatrix} A(\varepsilon U_{\Lambda,1}) & 0 \\ 0 & A(\varepsilon \hat{U}_{\Lambda,2}) \end{pmatrix} \hat{V} + B_\Lambda \hat{V} \end{aligned}$$

and we obtain that

$$\frac{d}{dt}\hat{V} + \left[\begin{pmatrix} A(\varepsilon U_{\Lambda,1}) & 0 \\ 0 & A(\varepsilon \hat{U}_{\Lambda,2}) \end{pmatrix} + B_\Lambda \right] \hat{V} = G_\Lambda(\hat{V}).$$

Since $U_\Lambda \in S_q(\kappa)$, we have that

$$A(\varepsilon U_{\Lambda,1}), A(\varepsilon \hat{U}_{\Lambda,2}) \in \mathcal{H}(W_{q,D}^2(I), L_q(I); k, \omega)$$

with a spectral bound less than $-\omega < 0$. Since

$$\|B_\Lambda\|_{\mathcal{L}(W_{q,D}^2(I) \times W_{q,D}^2(I), L_q(I) \times L_q(I))} \rightarrow 0, \quad \Lambda \rightarrow 0,$$

the operator $-(\text{diag}(A(\varepsilon U_{\Lambda,1}), A(\varepsilon \hat{U}_{\Lambda,2})) + B_\Lambda)$ generates an analytic semigroup on $L_q(I) \times L_q(I)$ with a negative spectral bound, cf. [9] for more details in a similar case. Then we can apply [30, Theorem 9.1.2] to conclude Theorem 5. From Theorem 5.(ii) and the Lipschitz continuity of $\tilde{\varphi}$ obtained in [25, Proposition 1], we also conclude that

$$\|\tilde{\varphi}_{u,v} - \Phi_\Lambda\|_{W_2^2(\Omega)} \leq R' e^{-\omega_0 t} \|(u_0, v_0) - U_\Lambda\|_{W_{q,D}^2(I) \times W_{q,D}^2(I)}, \quad \forall t \geq 0.$$

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DR. MARTIN KOHLMANN, GOERDELERSTRAE 36, 38228 SALZGITTER, GERMANY
E-mail address: `martin_kohlmann@web.de`